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J. F. A.
Simpson

(Simpson)

[illegible]

# THE
# DOCTRINE AND APPLICATION
# OF
# FLUXIONS.

CONTAINING
(BESIDES WHAT IS COMMON ON THE SUBJECT)
A NUMBER OF
*NEW IMPROVEMENTS IN THE THEORY;*
AND
THE SOLUTION OF A VARIETY OF NEW AND VERY INTERESTING
PROBLEMS,
IN DIFFERENT BRANCHES OF
THE MATHEMATICS.

By THOMAS SIMPSON, F.R.S.

IN TWO VOLUMES:—VOL. I.

*A NEW EDITION,*
CAREFULLY REVISED, AND ADAPTED, BY
COPIOUS APPENDIXES,
TO THE PRESENT ADVANCED STATE OF SCIENCE.
BY A GRADUATE OF THE UNIVERSITY OF CAMBRIDGE.

*LONDON:*
PRINTED FOR J. COLLINGWOOD, AND G. & W. B. WHITTAKER;
AND SOLD BY MESSRS. BELL & BRADFUTE, EDINBURGH.

1823.

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# ADVERTISEMENT.

THE Mathematics in general, and particularly the Fluxionary and Integral Calculus, having, for many years, been greatly progressive, it would have been a great omission on the part of the editor of a work of such standard excellence as SIMPSON'S FLUXIONS, to have entirely overlooked the obvious improvements. It has, therefore, been the primary object in this new edition, in addition to a corrected copy of the text, to furnish the Student with the most elementary theories, on a variety of new and interesting subjects—to present him, as it were, with the links, connecting past with present science. If the labours of the editor effect, in the smallest degree, a closer union between what have been too long separated, he will have the satisfaction of feeling they were not made in vain.

To the *Student*, any explanation at present with regard to the Appendixes would be unintelligible, and to the *Adept*, unnecessary.

An author of Simpson's celebrity needs no recommendation. The ablest Geometers of this and other countries, have at all times bestowed upon him the most liberal praise. The following extracts will serve as specimens:

" Demonstrationes olim contrahere studui ut auditorum meorum comodo consulerem et Theorian Fluxionum paucis exhibere: quod fere eodem tempore in Anglia præstitit Thomas Simpson."

*Frisius, Vol. 1, page* 287.

" Thomas Simpson, the ablest Analyst (if we regard the useful purposes of Analytical Science) that this country can boast of, &c."

*Woodhouse, Phys. Ast. page* 202.

" At the moment when we now write, the treatises of Maclaurin and Simpson are the best which we have on the Fluxionary Calculus, &c."

*Playfair's Works, Vol.* 4, *page* 323.

The last quoted author, however, proceeds to state, that excellent as these treatises are, they are still inadequate to the present state of science.

*January*, 1823.

TO THE

RIGHT HONOURABLE

# GEORGE EARL OF MACCLESFIELD.

&c. &c. &c.

MY LORD;

AS I esteem it a very great honour to be permitted to place the following sheets under your Lordship's protection, who is not only an encourager of, but an ornament to, mathematical learning; I have taken more than ordinary pains, that, what is here ushered into the world, with such advantage, may not be found altogether unworthy of so distinguished a patron.

I am not vain enough to imagine, that, to one so deeply read in these abstruse and curious speculations, as your Lordship is universally allowed to be, this work will appear without faults: but, then, I have the satisfaction to think, on the other hand, that, whatever is here to be met with capable of bearing the test of an exact and solid judgment, will also have its due weight, and not fail of receiving your Lordship's approbation: and if, upon the whole, there is merit enough found to entitle me to a favourable reception, it will gratify the highest ambition of,

My Lord,

Your Lordship's

Most Obedient Humble Servant,

THOMAS SIMPSON.

# THE AUTHOR'S PREFACE.

HAVING, in the year 1737, published a piece, on this same subject, under the title of *A Treatise of Fluxions* (whereof the whole impression hath been long since sold), it may be proper here, first of all, to assign the reasons why this work is sent abroad into the world as a new book, rather than a second edition of the said Treatise. Which, in short, are these two: First, because the present work is vastly more full and comprehensive; and, secondly, because the principal matters in it which are also to be met with in that Treatise, are handled in a different manner.

Besides the press-errors with which the said Treatise abounds, there are several obscurities and defects (which the Author's want of experience, and the many disadvantages he then laboured under, in his first sally, may, it is hoped, in some measure excuse). But what is now offered to the public, being a performance of more mature consideration and judgment, it will, I flatter myself, be found much more correct, and claim a favourable reception ; especially, as particular care and pains have been taken to put every thing in a clear light, and to oblige the lower, as well as the more experienced, class of readers.

The notion and explication *here* given of the first principles of fluxions, are not essentially different from what they are in the above-mentioned Treatise, though expressed in other terms. The consideration of time, which I have introduced into the general definition, will, perhaps, be disliked by *those* who would have fluxions to be *mere velocities :* but the advantage of considering them *otherwise* (not as the velocities themselves, but the magnitudes *they* would uniformly generate in a given finite time), appear to me sufficient to obviate any objection on that head.

By taking fluxions as *mere velocities*, the imagination is confined, as it were, to a point, and, without proper care, insensibly involved in metaphysical difficulties : but according to our method of conceiving and explaining the matter, less caution in the learner is necessary, and the higher orders of fluxions are rendered much more easy and intelligible. Besides, though Sir Isaac Newton defines fluxions to be *the velocities of motions*, yet he hath recourse to the increments, or moments, generated in equal particles of time, in order to determine those velocities ; which he afterwards teaches us to expound by finite magnitudes of other kinds: without which (as is already hinted above) we could have but very obscure ideas of the higher orders of fluxions: for if motion in (or at) a point be so difficult to conceive, that *some* have even gone so far as to dispute the very existence of motion, how much more perplexing must it be to form a conception, not only, of the velocity of a motion, but also infinite changes and affections of *it*, in one and the same point, where all the orders of fluxions are to be considered.

Seeing the notion of a fluxion, according to our manner of defining it, supposes an uniform motion, it may, perhaps, seem a matter of difficulty, at first view, how the fluxions of quantities, generated by means of accelerated and retarded motions, can be rightly assigned; since not any, the least, time can be taken during which the generating celerity continues the same: here, indeed, we cannot express the fluxion by any increment or space, *actually*, generated in a given time (as in uniform motions). But, then, we can easily determine, what the contemporary increment, or generated space *would be*, if the acceleration, or retardation, was to cease at the proposed position in which the fluxion is to be found: whence the true fluxion, itself, will be obtained, without the assistance of infinitely small quantities, or any metaphysical considerations.

Thus, for example, the motion of a ball, descending by the force of its own gravity, is continually accelerated; but to have the fluxion of the distance fallen through at any given position of the ball, we must find how far the ball *would*, uniformly, descend, from that point, in a given time, if the gravity, or the earth's attraction, from thence, was to cease acting. By which means we shall have as clear an idea of the fluxion and the true measure of the velocity of the ball, at any point assigned, as in those cases where the motion is, *actually*, uniform.

Again, if a right-line be supposed to move parallel to itself with an equable motion, and to increase in length, at the same time; the area generated thereby, will increase with an accelerated velocity; but the fluxion thereof, at any given position of the line, will be had by taking that part of the increment which *would*, uniformly, arise, was the length (as well as the velocity) of the line to continue invariable from the proposed position. For, if the length be supposed to increase, from the said position, the area generated, from thence, will be, evidently, greater than that which would uniformly arise in the same time; since the new parts, produced each succeeding moment, are greater and greater. Therefore the fluxion must be less than any space that can be described, in the given time, when the line increases. And, in the same manner, the fluxion will appear to be greater than any space that can be described, in the same time, when the line decreases. *It* must, therefore, be equal to that space, which will arise, when the length of the generating line, from the given position, is supposed neither to increase nor decrease: agreeable to Art. 4.

Thus much it seemed proper to offer here with regard to the first principles. I shall now proceed to say something concerning the order observed in treating, and putting together, the several parts of the work; wherein the ease and benefit of the young beginner have been particularly consulted: to load such an one with a multitude of rules and precepts, before giving him any taste of their use and application, would, certainly, be very discouraging; and like obliging a traveller to ascend an high mountain, without allowing him to stop by the way, to take breath, and refresh his spirits with a prospect of the agreeable and extensive view he has to expect when he arrives at the summit: I have, therefore, after demonstrating the first principles, proceeded immediately to exemplify their utility in several entertaining inquiries, before touching at all upon the inverse method, or the more difficult parts of the direct. And, since that branch of the inverse method which treats of the comparison of fluents is, naturally, somewhat difficult, it is referred to the second part of the work, together with such other matters in the Theory as might appear, either too tedious or hard to a learner at first setting out. The like care has been taken in the disposal of the rest of the work. As to the several particulars whereof *it* is composed, I must refer to the book itself, they being too many to be here enumerated: one thing, however, I must not omit to take notice of, relating to that part which treats of the aforesaid business of fluents; to which it may, perhaps, be objected, that notwithstanding my having insisted so largely on the subject, there are a number of forms of fluxions and fluents to be met with in authors, that I have not so much as touched upon. This is granted; but then they are most of them such as, I dare pronounce, can never arise in any inquiry into nature; and it would, doubtless, be time and labour misapplied, to swell the work, and embarrass the learner with a number of unnecessary difficulties, and empty speculations; when what is, really, proper and useful, in the subject, is sufficient (it is well known), to exercise his utmost attention and resolution.

I cannot put an end to this Preface without acknowledging my obligations to a small tract, entitled, *An Explanation of Fluxions in a Short Essay on the Theory*; printed for W. Innys: wrote by a worthy friend of mine (who was too modest to put his name to that, his first attempt) whose manner of determining the fluxion of a rectangle, and illustrating the higher orders of fluxions, I have, in particular, followed, with little or no variation.

# THE

# DOCTRINE AND APPLICATION

# OF

# FLUXIONS.

## *PART THE FIRST.*

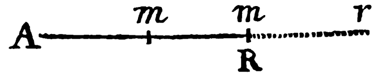
## SECTION I.

### *Of the Nature and Investigation of Fluxions.*

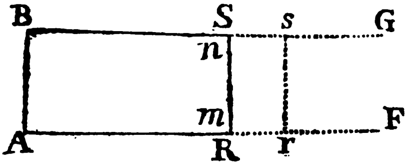
1. IN order to form a proper idea of the nature of Fluxions, all kinds of magnitudes are to be considered as generated by the *continual* motion of some of their bounds or extremes; as a Line by the motion of a Point; a Surface by the motion of a Line; and a Solid by the motion of a Surface.

2. Every quantity so generated is called a variable, or flowing quantity: *and the magnitude by which any flowing quantity* WOULD BE *uniformly increased in a given portion of time, with the generating celerity at any proposed position, or instant (was it from thence to continue invariable) is the fluxion of the said quantity at that position, or instant.*

Thus, let the point *m* be conceived to move from A, and generate the variable right-line A *m*, by a motion any how regulated; and let the celerity thereof, when it arrives at any proposed position R, be such *as would*, was it to continue uniform from that point, be sufficient to describe the distance, or line R *r*, in the given time allotted for the fluxion: then will R *r* be the fluxion of the variable line A *m*, in that position.

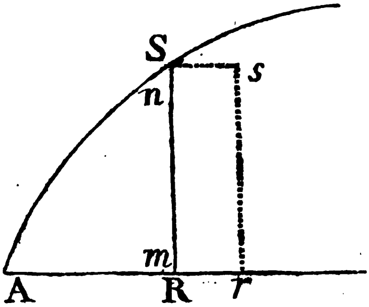


3. The fluxion of a plane surface is conceived in like manner, by supposing a given right-line *m n* to move parallel to itself, in the plane of the parallel, and immoveable lines A F and B G: for if (as above) R r be taken to express the fluxion of the line A *m*, and the rectangle R r s S be completed, then that rectangle, being the space which *would be* uniformly described by the generating line *m n*, in the time that A *m would be* uniformly increased by *m* r, is therefore the fluxion of the generated rectangle B *m*, in that position, according to the true meaning of the definition.



4. If the length of the generating line *m n* continually varies, the fluxion of the area will *still* be expounded by a rectangle under that line and the fluxion of the abscissa, or base: for, let the curvilineal space A *m n* be generated by the continual and parallel motion of the (now) variable line *m n*, and let R r be the fluxion of the base, or abscissa, A *m* (as before); then the rectangle R r s S will here also be the fluxion of the generated space A *m n*: because, if the length and velocity of the generating line *m n* were

to continue invariable from the position R S, the rectangle R$rs$S would then be uniformly generated, with the very celerity wherewith it begins to be generated, or with which the space A$mn$ is increased in that position.



5. From what has been hitherto said, it will appear, that *the fluxions of quantities are always as the celerities by which the quantities themselves increase in magnitude:* whence it will not be difficult to form a notion of the fluxions of quantities otherwise generated; as well such as arise from the revolution of right-lines and planes, as those by parallel motion: but of this hereafter. I come now to show the manner of determining the fluxions of algebraic quantities; by which all others, of what kind soever, are explicable. But first of all it will be requisite to premise the following observations.

I. *That the final letters* u, w, x, y, z, *of the alphabet are commonly put for variable quantities; and the initial letters* a, b, c, d, &c. *for invariable ones:* thus the diameter of a given circle may be denoted by $a$, and the sine of any arch thereof (considered as variable) by $x$.

II. *That the fluxion of a quantity represented by a single letter, is usually expressed by the same letter with a dot or full-point over it:* thus the fluxion of $x$ is represented by $\dot{x}$, and that of $y$ by $\dot{y}$.

III. *That the fluxion of a quantity which decreases, instead of increasing, is to be considered as negative.*

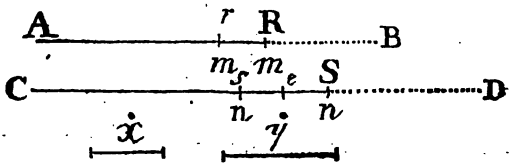
## PROPOSITION I.

6. *The Fluxion of a Quantity being given, it is proposed to find the Fluxion of any Power of that Quantity.*

As a clear understanding of this problem will be of great importance throughout the whole work, it may not be improper to consider it first in one or two of its most simple cases.

*Case* 1. Let $\dot{x}$ express the fluxion of $x$, (according to the foregoing notation) and let the fluxion of $x^2$ be required.

Conceive two points, $m$ and $n$, to proceed, at the same time, from two other points, A and C, along the right-lines A B and C D, in such sort, that the measure of the distance C S ($y$), described by the latter, may be, *always*, equal to the square of that A R ($x$), described by the former moving uniformly.



Furthermore, let $r$, $s$ and R, S be any contemporary positions of the generating points, and let the lines $\dot{x}$ and $\dot{y}$ represent the respective distances that *would be* uniformly described, in the same time, with the celerities of those points at R and S, then those lines will express the fluxions of A $m$ and C $n$ in this position, *(by the Definition,* Art. 2 and 5).

Moreover, since $Cs = Ar^2$ and $CS = AR^2$ *(by hypothesis)*, if R $r$ be denoted by $v$, we shall have C S ($y$) $= x^2$, and C $s$ ($= \overline{x-v}|^2$) $= x^2 - 2xv + v^2$, and consequently S $s$ ($= CS - Cs$) $= 2xv - v^2$; from whence we gather, that while the point $m$ moves over the distance $v$, the point $n$ moves over the distance

$2xv - v^2$. But this last distance (since the square of any quantity is known to increase faster, in proportion, than the root) is not described with an uniform motion (like the former), but an accelerated one; and therefore is equal to, and may be taken to express, the uniform space that might be described with the mean celerity at some intermediate point $e$, in the same time. Therefore, seeing the distances that might be described, in equal times, with the uniform celerity of $m$, and the mean celerity at $e$, are to each other as $v$ to $2xv - v^2$, or as 1 to $2x - v$, or, lastly, as $\dot{x}$ to $2x\dot{x} - v\dot{x}$, (all of which are in the same proportion) it is evident, that in the time the point $m$ would move uniformly over the distance $\dot{x}$, the other point $n$, with its celerity at $e$, *would* move uniformly over the distance $2x\dot{x} - v\dot{x}$. This being the case, let $r$, R, and $s$, S, be now supposed to coincide, by the arrival of the generating points at R and S, then $e$ (being always between $s$ and S) will likewise coincide with S; and the distance, $2x\dot{x} - v\dot{x}$, which might be uniformly described in the aforesaid time, with the velocity at $e$ (now at S), will become barely equal to $2x\dot{x}$; which *(by the Defin.)* is equal to $(\dot{y})$, the true fluxion of C $n$ or $x^2$.*

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* It may, perhaps, seem inaccurate, that the fluxions of $x$ and $x^2$ are compared together, and expressed both by lines, when the flowing quantities themselves, considered as a right-line and a square, admit of no comparison. This objection would, indeed, be of force, were the expressions restrained to a geometrical signification; but here our notions are more abstracted and universal, not obliging us to regard what kind of extension, may be defined by this or that expression, but only the values of the algebraic quantities thereby signified; to which the measures of all other quantities whatever are ultimately referred. And though quantities of different kinds cannot be compared with each other, their measures, in numbers, may. Thus, for instance, though it would be wrong to affirm, that a square whose area is 9 inches, is equal to a line of 9 inches long, yet it is no impropriety at all to say the numbers expressing their measures, in inches, are equal.

7. *Case* 2. Let the fluxion of $x^3$ be required.

Suppose every thing to remain as in the preceding case; only let C $n$ be here equal to the cube of A $m$ (instead of the square).

Then, in the very same manner, we have S $s$ $(=\mathrm{C\,S}-\mathrm{C}\,s=x^3-\overline{x-v}|^3)=3x^2v-3xv^2+v^3$: from whence it appears, that the distances which *might be* described in the same time, with the uniform celerity of $m$, and the mean celerity at $e$, will in this case, be to each other as $v$ to $3x^2v-3xv^2+v^3$, or as $\dot{x}$ to $3x^2\dot{x}-3xv\dot{x}+v^2\dot{x}$: which last expression, when $s$, $e$, and S coincide (as before) will become $3x^2\dot{x}$, the true fluxion of $x^3$ required.

8. *Universally*. Let C $n$ be *always* equal to $\overline{\mathrm{A}\,m}|^n$; also let $\overline{x-v}|^n$ (or $x-v$ raised to the power whose exponent is $n$) be represented by $x^n-ax^{n-1}v+bx^{n-2}v^2-cx^{n-3}v^3$, *&c.* and let every thing else be supposed as above.

Then, since S $s$ $(x^n-\overline{x-v}|^n)$ is $=ax^{n-1}v-bx^{n-2}v^2+cx^{n-3}v^3$, *&c.* it is plain that the spaces which might be described in the same time, with the uniform celerity of $m$, and the mean celerity at $e$, will here be to each other as $v$ to $ax^{n-1}v-bx^{n-2}v^2+cx^{n-3}v^3$, *&c.* or as $\dot{x}$ to $ax^{n-1}\dot{x}-bx^{n-2}v\dot{x}+cx^{n-3}v^2\dot{x}$, *&c.*

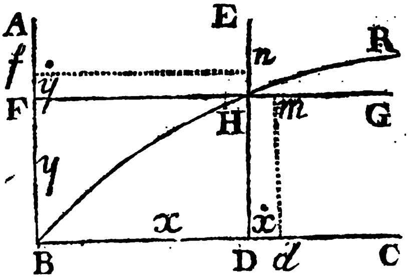
Therefore, all the terms wherein $v$ is found, vanishing, when $s$, $e$, and S coincide, we have $ax^{n-1}\dot{x}$ for the required fluxion of C$n$, or $x^n$; which fluxion, because the numeral co-efficient of the second term of a binomial involved is known to be, *universally*, equal to the exponent of the power, will also be truly expressed by $nx^{n-1}\dot{x}$. Q. E. I.

9. If the quantity A $m$ (or $x$) be generated with an accelerated, or a retarded motion, instead of an uniform one, the fluxion of $x^n$ (or C$n$) will come out exactly the same:

For the spaces $r$ R and $s$ S, actually described in the same time, being always, to each other, in the ratio of $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}v\dot{x}$, &c. the mean celerities, at certain intermediate points between $r$, R, and $s$, S, must also be in that ratio: which, when $v$ vanishes (as above) will become that of $\dot{x}$ to $ax^{n-1}\dot{x}$, (or $nx^{n-1}\dot{x}$) the very same as before.

## PROPOSITION II.

### 10. *To find the Fluxion of the Product or Rectangle of two variable Quantities.*



Conceive two right-lines D E and F G, perpendicular to each other, to move from two other right-lines, B A and B C, continually parallel to themselves, and thereby generate the rectangle DF. Let the path of their intersection, or the loci of the angle H, be the line B H R; also let D $d$ ($\dot{x}$) and F $f$ ($\dot{y}$), be the fluxions of the sides B D ($x$) and B F ($y$), and let $dm$ and $fn$, parallel to D H and F H be drawn. Therefore, because the fluxion of the space or area B D H is truly expressed by the rectangle D $m$ ($= y\dot{x}$\*) and that of the area, or space B F H, by the rectangle F $n$, and equal quantities have equal fluxions, it follows that the fluxion of the rectangle $xy =$ D F ($=$ B D H $+$ B F H) is truly expressed by $y\dot{x} + x\dot{y}$. Q. E. I.

* Art. 4.

### *The same otherwise.*

11. Let $xy$ be the given rectangle (as before); and put $z = x + y$, then $z^2$ being $= x^2 + 2xy + y^2$, we have $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 = xy$. But the fluxion of $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$ (and consequently that of its equal $xy$), is $z\dot{z} - x\dot{x} - y\dot{y}$ (by *Art.* 6): which, because $z = x + y$ and $\dot{z} = \dot{x} + \dot{y}$, is also equal to $\overline{x+y} \times \overline{\dot{x}+\dot{y}} - x\dot{x} - y\dot{y} = y\dot{x} + x\dot{y}$. Q. E. I.

### COROLLARY 1.

12. Hence the fluxion of the product of three variable quantities $(yzu)$ may be derived: for, if $x$ be put $= zu$, then $yzu$ will become $= yx$, and its fluxion $= y\dot{x} + y\dot{x}$ *(as above)*: but $x$ being $= zu$, and therefore $\dot{x} = z\dot{u} + u\dot{z}$, if these values be substituted in $x\dot{y} + \dot{x}y$, it will become $y \times \overline{\dot{z}u + \dot{u}z} + zu\dot{y} = yz\dot{u} + yu\dot{z} + zu\dot{y}$, the fluxion of $yzu$ required. In like manner, the fluxion of $xyzu$ will appear to be $xyz\dot{u} + xy\dot{z}u + \dot{x}yzu + x\dot{y}zu$, and that of $xyzuw = xyzu\dot{w} + xyz\dot{u}w + xy\dot{z}uw + x\dot{y}zuw + \dot{x}yzuw$.

### COROLLARY 2.

13. Hence, also, the fluxion of a fraction $\frac{u}{z}$ may be determined. For, putting $x = \frac{u}{z}$, we have $xz = u$, and therefore $x\dot{z} + z\dot{x} = \dot{u}$ *(as above)*; whence, by transposition and division, $\dot{x} = \frac{\dot{u}}{z} - \frac{x\dot{z}}{z} = \frac{\dot{u}}{z} - \frac{u\dot{z}}{z^2}$ (by writing $\frac{u}{z}$ for $x$) $= \frac{z\dot{u} - u\dot{z}}{z^2}$; which is the true fluxion of $x$, or its equal $\frac{u}{z}$, the fraction proposed.

14. Now, from the foregoing propositions, and their subsequent corollaries, the following practical rules

for determining the fluxions of algebraic quantities, are obtained.

## RULE I.

To find the fluxion of any given power of a variable quantity.

*Multiply the fluxion of the root by the exponent of the power, and the product by that power of the same root whose exponent is less by unity than the given exponent.*

This rule is investigated in Prop. 1, and is nothing more than $nx^{n-1}\dot{x}$ (the fluxion of $x^n$) expressed in words.

Hence the fluxion of $x^3$ is $3x^2\dot{x}$; that of $x^5$ is $5x^4\dot{x}$; and that of $\overline{a+y}|^7$ is $7\dot{y}\times\overline{a+y}|^6$, (because $a$ being constant, $\dot{y}$ is the true fluxion of the root $a+y$, in this case).

Moreover the fluxion of $\overline{a^2+z^2}|^{\frac{3}{2}}$, will be $\frac{3}{2}\times 2z\dot{z}\times\overline{a^2+z^2}|^{\frac{1}{2}}$, or $3z\dot{z}\sqrt{a^2+z^2}$: for here, $x$ being put $=a^2+z^2$, we have $\dot{x}=2z\dot{z}$, and therefore $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, the fluxion of $x^{\frac{3}{2}}$ (or $\overline{a^2+z^2}|^{\frac{3}{2}}$) is $=3z\dot{z}\sqrt{a^2+z^2}$, as above.

## RULE II.

15. To find the fluxion of the product of several variable quantities multiplied together.

*Multiply the fluxion of each by the product of the rest of the quantities, and the sum of the products thus arising will be the fluxion sought.**

Thus the fluxion of $xy$ is $x\dot{y}+y\dot{x}$; that of $xyz$ is $xy\dot{z}+xz\dot{y}+yz\dot{x}$; and that of $xyzu$, is $xyz\dot{u}+xyu\dot{z}+xzu\dot{y}+yzu\dot{x}$.

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* Art. 12.

## RULE III.

16. To find the fluxion of a fraction.

*From the fluxion of the numerator drawn into the denominator, subtract the fluxion of the denominator drawn into the numerator, and divide the remainder by the square of the denominator.**

Thus, the fluxion of $\frac{x}{y}$ is $\frac{y\dot{x}-x\dot{y}}{y^2}$; that of $\frac{x}{x+y}$, is $\frac{\dot{x}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times x}{\overline{x+y}|^2} = \frac{y\dot{x}-x\dot{y}}{\overline{x+y}|^2}$; and that of $\frac{x+y+z}{x+y}$, or $1+\frac{z}{x+y}$, is $\frac{\dot{z}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times z}{\overline{x+y}|^2}$; and so of others.

17. In the examples hitherto given, each is resolved by its own particular rule; but in those that follow, the use of two, and sometimes of all the three rules is requisite.

Thus (*by Rules* 1 *and* 2) the fluxion of $x^2y^2$ is $2x^2y\dot{y}+2y^2x\dot{x}$; that of $\frac{x^2}{y^2}$ is $\frac{2y^2x\dot{x}-2x^2y\dot{y}}{y^4}$, (*by Rule* 1 *and* 3) and that of $\frac{x^2y^2}{z}$ is $\frac{\overline{2x^2y\dot{y}+2y^2x\dot{x}}\times z-x^2y^2\dot{z}}{z^2}$; where all the three rules are necessary.

When the proposed quantity is affected by a co-efficient, or constant multiplicator, the fluxion found as above, must be multiplied by that co-efficient or multiplicator.

Thus, the fluxion of $5x^3$ is $15x^2\dot{x}$. For the fluxion of $x^3$ being $3x^2\dot{x}$, that of $5x^3$, which is 5 times as great, must consequently be $5\times 3x^2\dot{x}$, or $15x^2\dot{x}$.

And, in the very same manner the fluxion of $ax^n$ will appear to be $nax^{n-1}\dot{x}$. Moreover, the fluxion of $\frac{a}{\overline{x^2+y^2}|^{\frac{1}{2}}}$, or $a\times\overline{x^2+y^2}|^{-\frac{1}{2}}$, will be expressed by

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* Art. 13.

$a \times -\frac{1}{2} \times \overline{2x\dot{x}+2y\dot{y}} \times \overline{x^2+y^2}\big|^{-\frac{3}{2}}$, or $-\frac{a \times \overline{x\dot{x}+y\dot{y}}}{\overline{x^2+y^2}\big|^{\frac{3}{2}}}$; that of $\sqrt{\overline{x+y^{\frac{1}{2}}}}$, or $\overline{x+y^{\frac{1}{2}}}\big|^{\frac{1}{2}}$, by $\overline{\frac{1}{2}\dot{x}+\frac{1}{2}\times\frac{1}{2}\dot{y}y^{-\frac{1}{2}}} \times \overline{x+y^{\frac{1}{2}}}\big|^{-\frac{1}{2}}$, (*Rule* 1) or $\frac{\frac{1}{2}\dot{x}+\frac{1}{4}\dot{y}y^{-\frac{1}{2}}}{\sqrt{\overline{x+y^{\frac{1}{2}}}}}$, or $\frac{\frac{1}{2}\dot{x}y^{\frac{1}{2}}+\frac{1}{4}\dot{y}}{\sqrt{\overline{xy+y^{\frac{3}{2}}}}}$; and that of $\frac{\overline{x+a}\big|^2}{\sqrt{\overline{x^2-a^2}}}$ or $\frac{\overline{x+a}\big|^2}{\overline{x^2-a^2}\big|^{\frac{1}{2}}}$, by $\frac{2\dot{x} \times \overline{x+a} \times \overline{x^2-a^2}\big|^{\frac{1}{2}} - x\dot{x} \times \overline{x^2-a^2}\big|^{-\frac{1}{2}} \times \overline{x+a}\big|^2}{x^3-a^2}$; which by reduction, is $=\frac{2\dot{x} \times \overline{x^2-a^2}\big|^{\frac{1}{2}} - x\dot{x} \times \overline{x^2-a^2}\big|^{-\frac{1}{2}} \times \overline{x+a}}{x-a}$

$$=\frac{2\dot{x} \times \overline{x^2-a^2} - x\dot{x} \times \overline{x+a}}{\overline{x-a} \times \overline{x^2-a^2}\big|^{\frac{1}{2}}} = \frac{2\dot{x} \times \overline{x-a} \times \overline{x+a} - x\dot{x} \times \overline{x+a}}{\overline{x-a} \times \sqrt{\overline{x^2-a^2}}}$$

$$= \frac{\overline{x+a} \times \overline{x\dot{x}-2a\dot{x}}}{\overline{x-a} \times \sqrt{\overline{x^2-a^2}}}.$$

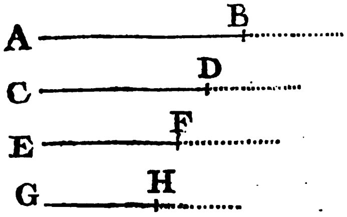
Having explained the manner of considering and determining the first fluxions of variable or flowing quantities, it will be proper to say something now concerning the higher orders, as second, third, fourth, &c. fluxions.

18. *The second fluxion of a Quantity is the fluxion of the variable or algebraic quantity expressing the first fluxion already defined.** *By the third fluxion is meant the fluxion of the variable quantity expressing the second: and by the fourth, the fluxion of the variable quantity expressing the third fluxion: and so on.*

Thus, for example, let the line A B represent a variable quantity, generated by the motion of the point B, and let the (first) fluxion thereof (or the space that *might be* uniformly described in a given time, with the celerity of B) be always expressed by the distance

* Art. 2.

of the point D from a given, or fixed point C: then if the celerity of B be not every where the same; the distance C D, expressing the measure of that celerity, must also vary, by the motion of D, from or towards C, according as the celerity of B is an increasing or a decreasing one: and the fluxion of the line C D, so varying (or the space (E F) that *might be* uniformly described in the aforesaid given time, with the celerity of D) is the second fluxion of A B. Again, if the motion of B be such that neither it, nor that of D (which depends upon it) be equable, then E F, expressing the celerity of D, will also have its fluxion G H; which is the third fluxion of A B, and the second fluxion of C D.



And thus are the fluxions of every other order to be considered, *being the measures of the velocities of which their respective flowing quantities, the fluxions of the preceding order are generated.**

19. Hence it appears, that a second fluxion always shows the rate of the increase or decrease of the first fluxion; and that third, fourth, &c. fluxions, differ in nothing (except their order and notation) from first fluxions, being actually such to the quantities from whence they are immediately derived; and therefore are also determinable, in the very same manner, by the general rules already delivered.

Thus, *by Rule* 3, the (first) fluxion of $x^3$ is $3x^2\dot{x}$: and, if $\dot{x}$ be supposed constant, that is, if the root $x$ be generated with an equable celerity, the fluxion of $3x^2\dot{x}$ (or $3\dot{x} \times x^2$) again taken, by the same rule, will be $3\dot{x} \times 2x\dot{x}$, or $6x\dot{x}^2$; which therefore is the second fluxion of $x^3$: whose fluxion, found in like sort, will be $6\dot{x}^3$, the third fluxion of $x^3$. Further than

*Art. 2.

which we cannot go in this case, because the last fluxion $6\dot{x}^3$ is here a constant quantity.

20. In the preceding example the root $x$ is supposed to be generated with an equal celerity: but if the celerity be an increasing or a decreasing one, then $\dot{x}$, expressing the measure thereof, being variable, will also have its fluxion; which is usually denoted by $\ddot{x}$: whose fluxion, according to the same method of notation, is again designed by $\dddot{x}$; and so on, with respect to the higher orders.

21. Here follow a few examples, wherein the root $x$ (or $y$) is supposed to be generated with a variable celerity.

Thus, the first fluxion of $x^3$ is $3x^2\dot{x}$ (or $3x^2 \times \dot{x}$). And, if the fluxion of $3x^2 \times \dot{x}$ (considered as a rectangle) be again found *(by Rule 2)*, we shall have $6x\dot{x} \times \dot{x} + 3x^2 \times \ddot{x} = 6x\dot{x}^2 + 3x^2\ddot{x}$, for the second fluxion of $x^3$.

Moreover, from the fluxion last found we shall in like manner get $6\dot{x} \times \dot{x}^2 + 6x \times 2\dot{x}\ddot{x} + 6x\dot{x} \times \ddot{x} + 3x^2 \times \dddot{x}$ (or $6\dot{x}^3 + 18x\dot{x}\ddot{x} + 3x^2\dddot{x}$) for the third fluxion of $x^3$.

Thus also, if $\dot{y} = nx^{n-1}\dot{x}$, then will $\ddot{y} = n \times \overline{n-1} \times x^{n-2}\dot{x}^2 + n\ddot{x}x^{n-1}$; and if $\dot{z}^2 = \dot{x}\dot{y}$, then will $2\dot{z}\ddot{z} = \dot{x}\ddot{y} + \dot{y}\ddot{x}$: and so of others. But, in the solution of problems, it will be convenient to make the first fluxion of some one of the simple quantities *(x or y)* invariable, not only to avoid trouble, but that it may serve as a standard to which the variable fluxions of the other quantities, depending thereon, may be always referred. The reader is also desired here (once for all) to take particular notice, that *the fluxions of all kinds and orders whatever, are contemporaneous, or such as may be generated together, with their respective celerities, in one and the same time.*

## SECTION II.

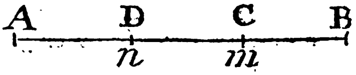
### *Of the Application of Fluxions to the Solution of Problems* DE MAXIMIS ET MINIMIS.

22. IF a quantity conceived to be generated by motion, increases or decreases till it arrives at a certain magnitude or position, and then, on the contrary, grows lesser or greater, and it be required to determine the said magnitude or position, the question is called a Problem *de Maximis et Minimis.*

### GENERAL ILLUSTRATION.

Let a point *m*, move uniformly in a right line, from A towards B, and let another point *n* move after it with a velocity either increasing, or decreasing, but so that it may, at a certain position, D, become equal to that of the former point *m*, moving uniformly.

This being premised, let the motion of *n* be first considered as an increasing one; in which case the distance of *n* behind *m* will continually increase, till the two points arrive at the cotemporary positions C and D; but afterwards it will again decrease; for the motion of *n*, till then, being slower than at D, it is also slower than that of the preceding point *m* (by hypothesis) but becoming quicker afterwards than that of *m*, the distance *m n* (as has been already said) will again decrease: and therefore is a *maximum*, or the greatest of all, when the celerities of the two points are equal to each other.



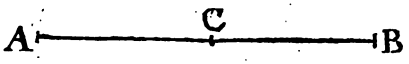
But, if *n* arrives at D with a decreasing celerity; then its motion being first swifter, and afterwards slower, than that of *m*, the distance *m n* will first decrease and then increase; and therefore is a *Minimum*, or the least of all, in the forementioned circumstance.

Since, then, the distance *m n* is a *maximum* or a *minimum*, when the velocities of *m* and *n* are equal, or when that distance increases as fast through the motion of *m*, as it decreases by that of *n*, its fluxion at that instant is evidently equal to nothing.* Therefore, as the motion of the points *m* and *n* may be conceived such that their distance *m n* may express the measure of any variable quantity whatever, it follows, that the fluxion of any variable quantity whatever, when a maximum or minimum, is equal to nothing.

* Art. 2 & 5.

## EXAMPLE I.

23. *To divide a given Right-line* A B *into two such Parts*, A C, B C, *that their Product, or Rectangle, may be the greatest possible.*



Put the given line A B $= a$, and let the part A C, considered as variable (by the motion of C from A towards B) be denoted by $x$: then B C being $= a - x$, we have A C × B C $= ax - x^2$: whose fluxion $a\dot{x} - 2x\dot{x}$ being put $= 0$, according to the prescript, we get $a\dot{x} = 2x\dot{x}$, and consequently $x = \frac{1}{2}a$. Therefore A C and B C, in the required circumstance, are equal to each other: which we also know from other principles.

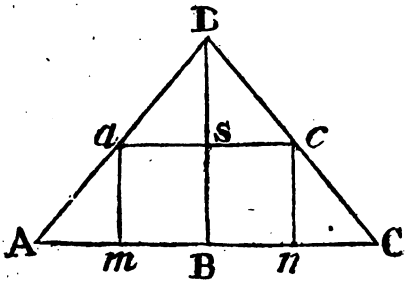
## EXAMPLE II.

24. *To find the Fraction which shall exceed its Cube by the greatest quantity possible.*

Let $x$ denote a variable quantity, expressing number in general; then the excess of $x$ above $x^3$ being universally represented by $x - x^3$, if the fluxion thereof be taken, &c. we shall have $\dot{x} - 3x^2\dot{x} = 0$; and therefore $x = \sqrt{\frac{1}{3}}$, the fraction required.

## EXAMPLE III.

25. *To determine the greatest Rectangle that can be inscribed in a given Triangle.*



Put the base A C of the given Triangle $= b$, and its Altitude B D $= a$; and let the altitude (B S) of the inscribed rectangle $m\,c$ (considered as variable) be denoted by $x$: Then, because of the parallel lines A C, and $ac$, it will be as B D $(a)$ : A C $(b)$ :: D S $(a-x)$ : $\frac{ba-bx}{a}$ $= ac$: whence the area of the rectangle, or $ac \times$ B S will be $= \frac{bax-bx^2}{a}$: whose fluxion $\frac{ba\dot{x}-2bx\dot{x}}{a}$ being (as before) put $=0$, we shall get $x = \frac{1}{2}a$. Whence the greatest inscribed rectangle is that whose altitude is just half the altitude of the triangle.

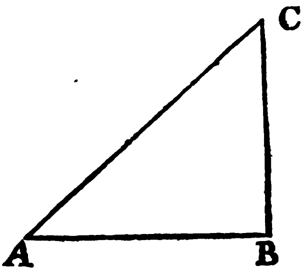
26. It will be proper to observe *here*, that the value of a quantity, when a *maximum* or *minimum*, may oftentimes be determined with more facility by taking the fluxion of some given part, multiple, or power thereof, than from the fluxion of the quantity itself. Thus, in the preceding example, where the general expression is $\frac{bax-bx^2}{a} = \frac{b}{a} \times \overline{ax-x^2}$, if the constant multiplicator $\frac{b}{a}$ be rejected, we shall have $ax - x^2$; whose fluxion $a\dot{x} - 2x\dot{x}$ being put $= 0$, we get $x = \frac{1}{2}a$, *the very same as before.*

The

The reason of which is obvious; because when the quantity itself (be it of what kind it will) is the greatest, or least possible, any *given* part, power, or multiple of it is also the greatest or least possible.

## EXAMPLE IV.

27. *Of all right-angled plane Triangles having the same given Hypothenuse, to find that* (A B C) *whose Area is the greatest.*



Let A C $=a$, A B $=x$, and B C $=y$: then, $x^2+y^2$ being $=a^2$, we shall have $y=\sqrt{a^2-x^2}$, and consequently $\frac{xy}{2} = \frac{x}{2}\sqrt{a^2-x^2}$ = the area of the triangle; whose square $\frac{a^2x^2}{4}-\frac{x^4}{4}$ being *also* a *maximum*,* the fluxion thereof $\frac{a^2x\dot{x}}{2} - x^3\dot{x}$ must therefore be $=0$ †: whence $x$ is found $=a\sqrt{\tfrac{1}{2}}$, and $y$ $(\sqrt{a^2-x^2}) = a\sqrt{\tfrac{1}{2}}$.

* Art. 26.

† Art. 22.

### *The same otherwise.*

Since $\frac{1}{2}xy$ is a *maximum*, and $x^2+y^2=a^2$, let the fluxions of both be taken, and you will have $\frac{1}{2}x\dot{y}+\frac{1}{2}y\dot{x}=0$, and $2x\dot{x}+2y\dot{y}=0$; from the former of which $\dot{y}$ will be $=-\frac{y\dot{x}}{x}$; and from the latter it will be $=-\frac{x\dot{x}}{y}$: Therefore $\frac{y\dot{x}}{x}$ and $\frac{x\dot{x}}{y}$ are equal to each other, and consequently $x=y$ (the same as before).

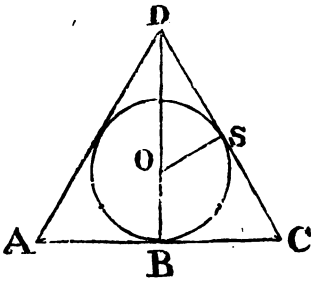
## EXAMPLE V.

28. *Of all right-angled plane Triangles containing the same given Area, to find that whereof the Sum of the two Legs* A B + B C *is the least possible.* (See the preceding figure.)

Let one leg, A B, be denoted by $x$, and the area of the triangle by $a$; then the other leg will be denoted by $\frac{2a}{x}$, and the sum of the two legs will be $x + \frac{2a}{x}$; whereof the fluxion is $\dot{x} - \frac{2a\dot{x}}{x^2}$; which, put $= 0$, gives $x$ (A B) $= \sqrt{2a}$: whence B C $\left(\frac{2a}{x}\right)$ is also $= \sqrt{2a}$. Therefore the two legs are equal to each other.

## EXAMPLE VI.

29. *To determine the Dimensions of the least Isosceles Triangle* ACD *that can circumscribe a given Circle.*



Let the Distance (O D) of the vertex of the triangle from the center of the circle, be denoted by $x$, and let the remaining part of the perpendicular, which is the radius of the circle, be represented by $a$: then, if O S, perpendicular to D C, be drawn, we shall have D S $= \sqrt{x^2 - a^2}$; and therefore since D S : O S :: D B : B C, we likewise have B C $= \frac{a \times \overline{x+a}}{\sqrt{x^2 - a^2}}$; which multiplied by $\overline{x+a}$ (BD) gives $\frac{a\times\overline{x+a}|^2}{\sqrt{x^2-a^2}}$ for the area of the triangle: which being a *minimum*, its square must be a *minimum*, and consequently $\frac{\overline{x+a}|^4}{x^2-a^2}$, or its equal $\frac{\overline{x+a}|^3}{x-a}$, a *minimum* also.* Whose fluxion, therefore, which is $\frac{3\dot{x}\times\overline{x+a}|^2\times\overline{x-a}-\dot{x}\times\overline{x+a}|^3}{\overline{x-a}|^2}$, being put $=0$, and the whole divided by $\frac{\dot{x}\times\overline{x+a}|^2}{\overline{x-a}|^2}$, we also get $3\times\overline{x-a}-\overline{x+a}=0$; whence $x=2a$: therefore, O D being $=$ 2 O S, and the triangles O D S and B D C equiangular, it is evident that D C is likewise $=$ 2 B C $=$ A C; and so the triangle A C D, when the least possible, is equilateral.

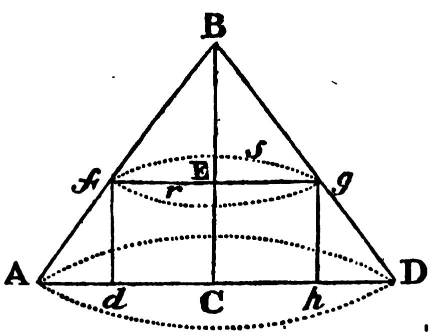
* Art. 26.

## EXAMPLE VII.

30. *To determine the greatest Cylinder, dg, that can be inscribed in a given Cone* A D B.

Let $a=$ B C, the altitude of the Cone;
$b=$ A D, the diameter of its base;
$x=fg$ $(dh)$ the diameter of the cylinder, considered as variable.
$p=\left(\frac{3,14159,\ \&c.}{4}\right)$ the area of the circle whose diameter is unity.

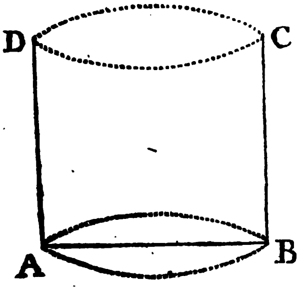
Then, the areas of circles being to one another as the squares of their diameters, we have $1^2 : x^2 :: p : (px^2)$ the area of the circle $fsgr$: moreover, from the similarity of the triangles A B C and A $df$, we have $\frac{1}{2}b$ (A C) $: a$ (B C) $:: \frac{1}{2}b-\frac{1}{2}x$ (A $d$) $: df = \frac{ab-ax}{b}$; which multiplied by the area $px^2$ (found above) gives



$\frac{pabx^2 - pax^3}{b}$ for the solid content of the cylinder: which being a *maximum*, its fluxion $\frac{2pabx\dot{x}}{b} - \frac{3pax^2\dot{x}}{b}$ must be $=0$,* consequently $x = \frac{2b}{3}$ and $df = \frac{a}{3}$: from whence it appears, that the inscribed cylinder will be the greatest possible, when the altitude thereof is just $\frac{1}{3}$ of the altitude of the whole cone.

## EXAMPLE VIII.

31. *To determine the Dimensions of a cylindric Measure* A B C D, *open at the top, which shall contain a given quantity (of Liquor, Grain, &c.) under the least internal Superficies possible.*



Let the diameter A B $=x$, and the altitude A D $=y$; moreover let $p$ (3,14159, &c.) denote the periphery of the circle whose diameter is unity, and let $c$ be the given content of the cylinder. Then it will be $1 : p :: x : (px)$ the circumference of the base; which, multiplied by/the altitude $y$, gives $pxy$ for the concave superficies of the cylinder. In like manner, the area of the base, by multiplying the same expression into $\frac{1}{4}$ of the diameter $x$, will be found $=\frac{px^2}{4}$; which drawn into the altitude $y$, gives $\frac{px^2y}{4}$ for the solid content of the cylinder; which being made $= c$, the concave surface $pxy$ will be found $=\frac{4c}{x}$, and consequently the whole surface $=\frac{4c}{x}+\frac{px^2}{4}$: whereof the fluxion, which is $-\frac{4c\dot{x}}{x^2}+\frac{px\dot{x}}{2}$, being put $= 0$, we shall get $-8c+px^3=0$; and therefore $x=2\sqrt[3]{\frac{c}{p}}$: further, because $px^3=8c$, and $px^2y=4c$, it follows, that $x=2y$; whence $y$ is also known, and from which it appears, that the diameter of the base must be just the double of the altitude.

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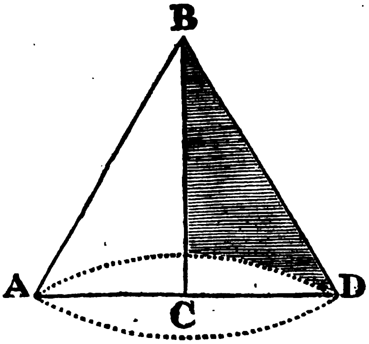
* Art. 22.

## EXAMPLE IX.

32. *Of all Cones under the same given Superficies* (s) *to find that* (ABD) *whose Solidity is the greatest.*

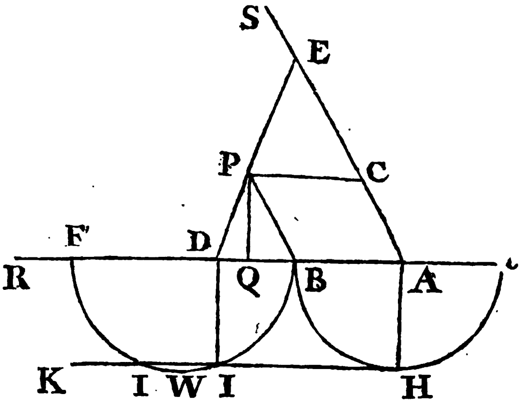
Let the semi-diameter of the base, $AC=x$, and the length of the slant side $AB=y$; and let $p$ (as in the preceding examples) denote the periphery of the circle whose diameter is unity.



Then the circumference of the base will be $= 2px$, the area of the base $= px^2$, and the convex superficies of the cone $= pxy$ (which last is found by multiplying half the periphery of the base by the length of the slant side): wherefore, since the whole superficies is $=px^2+pxy=s$, we have $y= \frac{s}{px}-x$; whence the altitude CB $(\sqrt{AB^2-AC^2}) = \sqrt{\frac{s^2}{p^2x^2}-\frac{2s}{p}}$; which multiplied by $\left(\frac{px^2}{3}\right)$ $\frac{1}{3}$ of the area of the base, gives $\frac{px^2}{3}\sqrt{\frac{s^2}{p^2x^2}-\frac{2s}{p}}$ for the solid content of the cone. Which being a *maximum*, its square $\frac{s^2x^2}{9}-\frac{2psx^4}{9}$ must also be a *maximum*; and therefore $\frac{2s^2x\dot{x}}{9}-\frac{8psx^3\dot{x}}{9}=0$; whence $s-4px^2=0$, and consequently $x=\sqrt{\frac{s}{4p}}$: from which $y$ $\left(= \frac{s}{px}-x=\frac{s-px^2}{px}=\frac{3px^2}{px}=3x\right)$ will likewise be known; and from whence it will appear that the greatest cone under a given surface (or a given cone under the least surface) will be when the length of the slant side is to the semi-diameter of the base in the ratio of 3 to 1, or (which comes to the same) when the square of the altitude is to the square of the whole diameter in the ratio of 2 to 1.

## EXAMPLE X.

33. *To determine the position of a Right-line* D E, *which, passing through a given point* P, *shall cut two Right-lines* A R *and* A S, *given by position, in such sort that the sum of the Segments,* A D *and* A E, *made thereby, may be the least possible.*



Make P B parallel to A S, $=a$, and P C, parallel to A R, $=b$; and let B D$=x$: then, by reason of the parallel lines, it will be, $x : a :: b : \mathrm{CE} = \frac{ab}{x}$: therefore A D+A E$=b+x+a+\frac{ab}{x}$, and its fluxion, $\dot{x}-\frac{ab\dot{x}}{x^2}$, which, in the required circumstance, being $=0$, we have $x^2-ab$ also $=0$, and consequently $x=\sqrt{ab}$; whence the position of D E is known. But the same thing may be otherwise determined, independent of fluxions, from the general solution of the problem for finding the position of D E, when the sum of the segments A D and A E (instead of being a *minimum*) shall be equal to a given quantity. Of which problem, the geometrical construction may be as follows.

Complete the parallelogram A B P C (as before) and in R A produced, take $Ac = AC$, and let $cF$ be equal to the given sum of the two segments: also let two semi-circles be described upon $Bc$ and $BF$, and let A H, perpendicular to $Bc$, intersect the former in H; likewise let H K, parallel to $Fc$, intersect the latter in I; draw I D perpendicular to $Fc$, and through P and D draw D E, which will be the position required. For $AB \times Ac$ being $= AH^2 = DI^2 = BD \times DF$, we have $BD : AB :: Ac\ (AC) : DF$; also, because of the parallel lines, we have $BD : AB :: AC : CE$; whence $DF = CE$, and consequently $AD + AE$ ($AD + AC + FD$) is equal to $cF$, which construction is more neat than that in p. 155 of my *Geometry*. But to show how far this may conduce to the matter first proposed, we are to observe, that as the problem here constructed appears to be impossible, when the right-line H K (instead of cutting or touching) falls wholly below the circle B W F, the least possible value of B F (and consequently of $AD + AE$) must therefore be when that right-line touches the circle; that is, when $BD = DI = AH = \sqrt{AB \times AC}$; which value is the very same with that found above.

The same conclusion may also be deduced from the algebraic solution of the aforesaid problem: for, putting $b + x + a + \frac{ab}{x}$ ($AD + AE$) $= s$, and solving the equation, $x$ will be found $= \frac{s-a-b}{2} \pm \sqrt{\frac{\overline{s-a-b}|^2}{4} - ab}$: which equation being no longer possible than till $\frac{\overline{s-a-b}|^2}{4} - ab$ is $= 0$, we have $x$, in that circumstance, $= \frac{s-a-b}{2} = \sqrt{ab}$; *still as before.* In like manner the *maxima* and *minima* may be determined in other cases, by finding the position or circumstance wherein the general problem begins to be impossible (supposing the quantity sought to be given). But the operation by fluxions is, for the general part, much more short and expeditious.

## EXAMPLE XI.

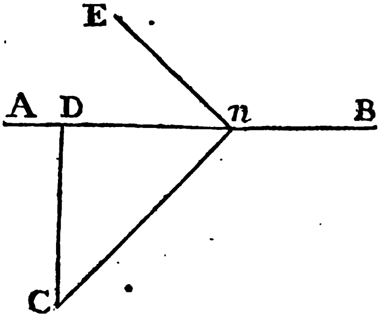
34. *The same being given as in the preceding Example, to determine the Position, when the Line* D E *itself is the least possible.*

Upon A F let fall the perpendicular P Q; make B Q $=c$, and the rest as before: then D P² being (= D B² + B P² − 2B Q × D B) $=x^2+a^2-2cx$, and D B² : D P² :: D A² : D E², we have $x^2 : x^2+a^2-2cx :: \overline{b+x}|^2$ : D E² $=\frac{\overline{b+x}|^2\times\overline{x^2-2cx+a^2}}{x^2}=\overline{b+x}|^2\times\overline{1-\frac{2c}{x}+\frac{a^2}{x^2}}$; whose fluxion, which is $2\dot{x}\times\overline{b+x}\times\overline{1-\frac{2c}{x}+\frac{a^2}{x^2}}+\overline{b+x}|^2\times\overline{\frac{2c\dot{x}}{x^2}-\frac{2a^2\dot{x}}{x^3}}$, being put $=0$, and the whole equation, divided by $2\dot{x}\times\overline{b+x}$, there will come out $1-\frac{2c}{x}+\frac{a^2}{x^2}+\overline{b+x}\times\overline{\frac{c}{x^2}-\frac{a^2}{x^3}}=0$; whence $x^3-2cx^2+a^2x+\overline{b+x}\times\overline{cx-a^2}=0$; that is (by reduction) $x^3-cx^2+bcx-a^2b=0$: from the resolution of which equation, the position of D E is determined.

## LEMMA.

35. *If a body or point* ($n$) *be supposed to move in a right-line* A B, *its absolute celerity, in the direction of that line, will be to the relative celerity, whereby it tends to or from a given point* C, *any where out of the line, as the distance* C$n$ *is to the distance* D$n$, *intercepted by* $n$ *and the perpendicular* C D; *or as radius to the co-sine of the angle of inclination* D $n$ C.

For, putting C D $=a$, D $n=x$, and C $n=y$, we have $a^2+x^2=y^2$, and consequently $2x\dot{x}=2y\dot{y}$ :* * Art. 2 & 5.



Whence $\dot{x} : \dot{y} :: y\ (Cn) : x\ (Dn) ::$ Radius : co-sine D*n*C: but the fluxions of quantities are as the celerities of their increase, * therefore the truth of the proposition is manifest.

*Art. 2 & 5.

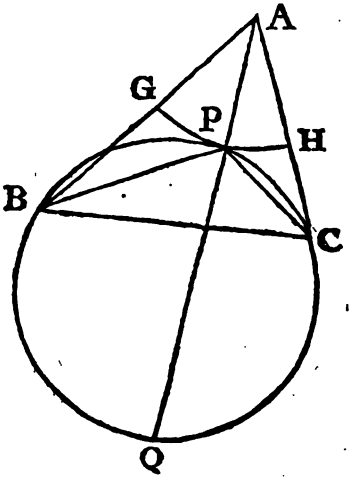
### COROLLARY.

It follows from hence, that the relative celerities in any two different directions, *n* E and *n* C, are directly as the co-sines of the corresponding angles D *n* E and D *n* C. Therefore, when *n* E is perpendicular to C *n*, (and the angle D *n* E therefore equal to C) the celerity in the direction *n* E, will be to that in the direction *n* C, as the sine of D *n* C is to its co-sine. From whence it appears, that the celerities in the directions D *n*, C *n*, and E *n* (perpendicular to *n* C) are to each other as C *n*, D *n*, and C D respectively.

## EXAMPLE XII.

36. *To determine the Position of a Point, from whence, if three Right-lines be drawn to so many given Points* A, B, C, *their Sum shall be the least possible.*

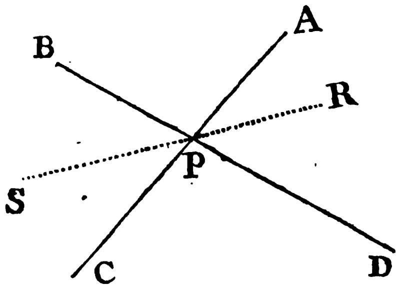
Let H P G be the periphery of a circle described about the point A, as a center, at any distance A G; in which let the point P be conceived to move with an uniform celerity, from G towards H. Then, because the relative celerity thereof, in the direction P C, is to that in the direction B P produced, as the co-sine of the angle C P H to the co-sine of the angle B P G *(by the preceding Lemma)*; and since these celerities, when the sum of C P and B P is a *minimum*, must be equal,* it follows, therefore, that the said angles C P H and B P G, as well as their co-sines, will in that circumstance become equal to each other; and consequently A P C also equal to A P B. From whence it appears, that (take A G what you will) the sum of the three lines, A P, B P, and C P, cannot be the least possible when the angles A P B and A P C are unequal. And, by the same argument, it also appears that their sum cannot be the least possible, when the angles B P A and B P C are unequal: therefore, their sum must be the least possible, when all the three angles about the point P are equal to one another; provided the case will admit of such an equality, or that no one of the angles of the triangle A B C is equal to, or greater than $\frac{1}{3}$ of 4 right angles (for otherwise the point P will fall in the obtuse angle): hence this



* Art. 2 & 22.

### CONSTRUCTION.

Describe, upon B C, a segment of a circle, to contain an angle of 120°, and let the whole circle B C Q be completed; and from A to the middle (Q) of the arch B Q C, draw A Q, intersecting the circumference of the circle in P; which will be the point required. For the angles B P Q and C P Q, standing upon the equal arches B Q and C Q, have their complements A P B and A P C equal to each other; and therefore the angle B P C being 120° (by construction) each of the said angles A P B, A P C, will likewise be 120 degrees.

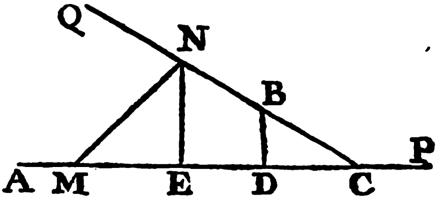


After the same manner, it will appear that the sum of all the lines A P, B P, C P, &c. drawn from any number of given points, A B C, &c. to meet in another point P, will be the least possible, when the co-sines of the angles R P A, R P B, R P C, &c. that the said lines make with any other line R S, passing through the point of concourse, destroy each other: which will be when all the angles A P B, B P C, C P D, &c. are equal in all cases where the position of the given points will admit of such an equality. But if the number of given points be four, the required point will be in the intersection of the two right-lines joining the opposite points: for supposing A P C and B P D to be continued right-lines, the cosine of R P A will be equal and contrary to that of R P C, and that of R P B equal and contrary to that of R P D.

## EXAMPLE XIII.

37. *If two Bodies move at the same Time, from two given Places* A *and* B, *and proceed uniformly from thence in given Directions,* A P *and* B Q, *with Celerities in a given Ratio; it is proposed to find their Position, and how far each has gone, when they are the nearest possible to each other.*

Let M and N be any two cotemporary positions of the bodies, and upon A P let fall the perpendiculars N E and B D; also let Q B be produced to meet A P

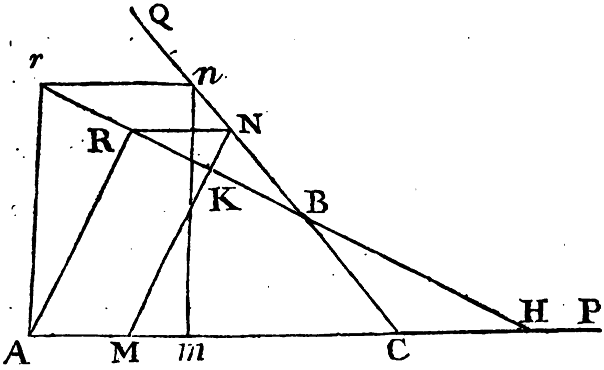


in C, and let M N be drawn: moreover, let the given celerity in B Q be to that in A P, as $n$ to $m$, and let A C, B C, and C D (which are also given) be denoted by $a$, $b$, and $c$, respectively, and make the variable distance C N$=x$: then, by reason of the parallel lines NE and BD, we shall have $b$ (CB) : $x$ (CN) :: $c$ (CD) : C E $= \frac{cx}{b}$. Also, because the distances, B N and A M, gone over in the same time, are as the celerities, we have likewise, $n : m :: x-b$ (B N) : A M $= \frac{mx-mb}{n}$, and consequently C M (A C $-$ A M) $= a + \frac{mb}{n} - \frac{mx}{n} = d - \frac{mx}{n}$ (by writing $d = a + \frac{mb}{n}$). Whence MN$^2$ ($=$CM$^2$+CN$^2$−CM × 2CE) will also be found $= \left(d - \frac{mx}{n}\right)^2 + x^2 - \overline{d - \frac{mx}{n}} \times \frac{2cx}{b} = d^2 - \frac{2dmx}{n} + \frac{m^2x^2}{n^2} + x^2 - \frac{2cdx}{b} + \frac{2cmx^2}{nb}$, whose fluxion $-\frac{2dm\dot{x}}{n} + \frac{2m^2x\dot{x}}{n^2} + 2x\dot{x} - \frac{2cd\dot{x}}{b} + \frac{4cmx\dot{x}}{nb}$ being made$=0$ (because MN is to be a *minimum*) we get $-bdmn + m^2bx + n^2bx - n^2cd + 2mncx = 0$; and consequently $x = \frac{mnbd + n^2cd}{m^2b + n^2b + 2mnc} = \frac{nd \times \overline{mb + nc}}{b \times \overline{m^2 + n^2} + 2mnc}$; from whence BN, A M, and M N are also given.

### *The same otherwise.*

Because the relative celerities of the two bodies, at M and N, in the direction of the line M N (produced) are truly expressed by $\frac{\textit{co-sine}\ \text{M}}{\textit{radius}} \times m$, and $\frac{\textit{co-s.}\ \text{N}}{\textit{rad.}}$
* Art. 35. $\times n$ respectively;* and as these celerities, when the distance M N is a *minimum*, do become equal to each
† Art. 22. other,† it follows that, in this circumstance, $m : n ::$ co-sine N. : co-sine. M :: secant of M : secant of N *(by plane Trig.)*

Whence this construction. Take C H to C B in the given ratio of $m$ to $n$, and draw H B; upon which

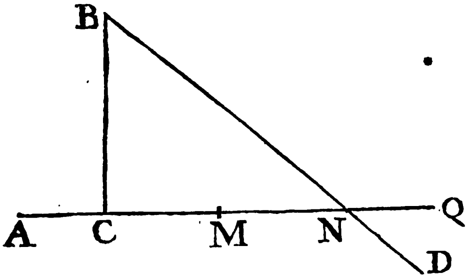


produced (if necessary) let fall the perpendicular A R; draw R N parallel to A H, meeting C Q in N; lastly, draw N M parallel to A R, and it will give the position required. For, first, it is plain, because A M (R N) : B N (:: C H : C B) :: $m : n$, that M and N are cotemporary positions: it is likewise plain, that R N and B N will be secants of the angles K N R (C M N) and K N B (C N M) to the radius N K; because the angle N K R (=A R K) is a right one. Which lines or secants are in the proposed ratio of $m$ to $n$, as has been already shown.

But the same solution may be, yet, otherwise derived, independent of fluxions, from principles entirely geometrical. For, let *m* and *n* be any two cotemporary positions at pleasure, and let C H (as before) be to C B, as the celerity in A P to that in C Q; moreover, let *n r*, parallel to A P, be drawn, meeting H B produced in *r*, and let A *r* be joined. Then, since C B : C H :: B *n* : *n r* (*by sim. triangles*) and C B : C H :: B *n* : A *m* (*by Hyp.*) it follows, that *n r* and A *m* (which are parallel) will also be equal to each other; and therefore A *r* and *m n*, likewise equal and parallel. But A *r* is the least possible when perpendicular to H *r*. Whence the solution is manifest.

## EXAMPLE XIV.

✓ 38. *Let the body* M *move uniformly from* A *towards* Q, *with the Celerity m, and let another Body* N *proceed from* B, *at the same time, with the Celerity n. Now it is proposed to find the Direction* (B D) *of the latter, so that the Distance* M N *of the two Bodies, when the latter arrives in the Way or Direction* A Q *of the former, may be the greatest possible.*



Let B C be perpendicular to A Q, and make A C $= a$, B C $= b$, and B N $= x$. Therefore, if the position M be supposed cotemporary with N, we shall have $n : m :: x : \text{AM} = \frac{mx}{n}$; whence $\text{CM} = \frac{mx}{n} - a$, and consequently MN (CN − CM) $= \sqrt{x^2 - b^2} - \frac{mx}{n} + a$; whereof the fluxion being taken, and made $= 0$, we get $\frac{x}{\sqrt{x^2 - b^2}} = \frac{m}{n}$; therefore $x = \frac{mb}{\sqrt{m^2 - n^2}}$, and CN $(\sqrt{x^2 - b^2}) = \frac{nb}{\sqrt{m^2 - n^2}}$: whence, $m : n ::$ BN : CN :: radius : co-sine N. The same conclusion is otherwise derived, thus,

Let the right-line BD be supposed to revolve about the point B as a center, with the motion so regulated, that the intercepted part thereof BN may increase with the uniform celerity $n$: then, the celerity with which CN is increased being $= \frac{n \times \textit{radius}^*}{\textit{co-sine}\ \mathrm{N}}$, this expression, when MN is a *maximum*, must consequently be equal to ($m$) the velocity of the other body† M; and therefore $m : n ::$ radius : co-sine N, as before.

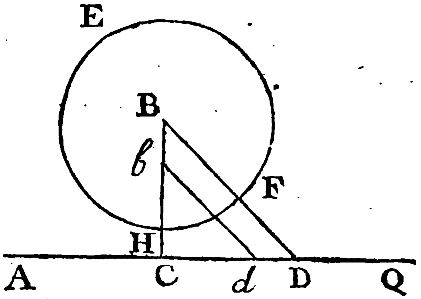
* Art. 35.

† Art. 22.

## EXAMPLE XV.

39. *Supposing a Ship to sail from a given place* A, *in a given Direction* AQ, *at the same time that a Boat, from another given place* B, *sets out in order (if possible) to come up with her, and supposing the rate at which each Vessel runs to be given; it is required to find in what Direction the latter must proceed, so that if it cannot come up with the former, it may, however, approach it as near as possible.*

Let the celerity of the ship be to that of the boat in the given ratio of $m$ to $n$; also let D and F be the places of the two vessels when nearest possible to each other, and from the center B, through F, suppose the circumference of a circle to be described. Then (the distance DF being the least possible) the point F must be in the right-line (DB) joining the point D and the



center B; because no other point in the whole periphery, at which the boat from B might arrive in the same time, is so near to D as that wherein the line DB intersects the said periphery. But now, to get an expression for D F, in algebraic terms, let B C be perpendicular to A Q, and make $AC=a$, $BC=b$, and $CD=x$; and then B D ($\sqrt{BC^2+CD^2}$) will be $=\sqrt{b^2+x^2}$; moreover, because $m:n::AD\ (a+x):BF$, you will have $BF=\frac{na+nx}{m}$, and consequently $DF=\sqrt{b^2+x^2}-\frac{na+nx}{m}$; whose fluxion, $\frac{x\dot{x}}{\sqrt{b^2+x^2}}-\frac{n\dot{x}}{m}$, being made $=0$, we find $x=\frac{nb}{\sqrt{m^2-n^2}}$; whence the direction B D is known: and, if the value of $x$, thus found, be substituted in that of D F (found above) we shall have $DF=\frac{b\sqrt{m^2-n^2}-na}{m}$; whence the position of F is known. And from which it is observable, that, as D F must be a *real*, *positive* quantity (by the question) this method of solution can only obtain when $m$ is greater than $n$, and $b\sqrt{m^2-n^2}$, also greater than $na$: for in all other cases the boat will be able to come up with the ship.

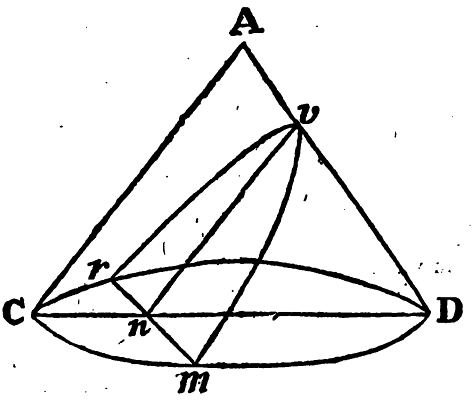
*The same otherwise.*

Let the radius of the circle E F H be conceived to increase uniformly, with the celerity $n$, whilst the point

D moves uniform along A Q, with the celerity $m$: then, the celerity at D, in the direction of B D produced, being $= \frac{m \times \textit{co-sine}\ D}{\textit{radius}}$, the relative celerity with which the point D recedes from the periphery of the said variable circle, will be universally expressed by $\frac{m \times \textit{co-sine}\ D}{\textit{radius}} - n$; which being $= 0$, when D F is a *minimum*, we have in this case $m \times \textit{co-sine}\ D = n \times \textit{radius}$, and consequently $m : n :: \textit{radius} : \textit{co-sine}\ D$. Therefore, if at C, a right-angled triangle C $b$ $d$ be constituted, whose base C $d = n$, and its hypothenuse $d\,b = m$, and parallel to the latter you draw B D, it will be the direction required: in which, if there be taken BF, a fourth proportional to $m$, $n$, and A D, you will also have the position required.

## EXAMPLE XVI.

40. *To determine the greatest Parabola that can be formed by cutting a given Cone* A C D.



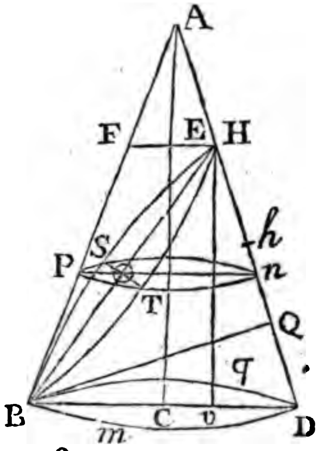
Let $n\,v$, parallel to C A, be the axis of the parabola $r\,v\,m$, and $r\,m$ the base (or ordinate) thereof; putting

$DC=a$, $CA=b$, and $Dn=x$; then, because of the parallel lines, it will be $a : b :: x : \frac{bx}{a} = nv$: moreover, by the property of the circle, we have $rn^2$ $(=nm^2=Dn \times Cn) = ax-x^2$, and consequently $rm$ $2\sqrt{ax-x^2}$; which, multiplied by $\frac{2}{3} \times \frac{bx}{a}$ (because every parabola is $\frac{2}{3}$ of a parallelogram of the same base and altitude) gives $\frac{4bx}{3a}\sqrt{ax-x^2}$ for the content of the parabola: whose fluxion, or that of $ax^3-x^4$ * being put equal to nothing, we find $x=\frac{3a}{4}$: whence $nv = \frac{3}{4} \times AC$, $rm = CD \times \sqrt{\frac{8}{4}}$, and the area of the greatest or required parabola $= AC \times CD \times \frac{\sqrt{3}}{4}$.

## EXAMPLE XVII.

41. *To determine the greatest Ellipsis* BTES *that can be formed by cutting a given Cone* ABD.

Let BE be the greater, and TS the lesser, axis of the ellipsis BTES, considered as variable by the motion of (the end of the transverse) E, along the line AD; moreover, let E$v$ be parallel to AC, the axis of the cone, meeting the diameter BD in $v$, and let the diameters EF and $np$ be parallel to BD, whereof the latter $np$ is supposed



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* Art. 26.

to pass through O, the centre of the ellipsis: then, putting A C $=a$, C D $=b$, and C $v=x$, we shall have B$v=b+x$; also, because of the parallel lines we have C D $(b)$ : C A $(a)$ : : D$v$ $(b-x)$ : $\frac{a\times\overline{b-x}}{b}=$ E$v$; whence

$$\text{B E } (\sqrt{\text{B}v^2+\text{E}v^2}) = \frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2}}{b}$$

Furthermore, since the triangles E O $n$, E B D, and B O $p$, B E F are equiangular, and E O $(=$B O$)=\frac{1}{2}$B E, we likewise have O $n=\frac{1}{2}$B D $=b$, and O p $=\frac{1}{2}$E F $=$ C $v=x$; and consequently O $n\times$ O $p$ ($=$O T$^2$, *by the property of the circle*) $=bx$; whence S T $=2\sqrt{bx}$, and

$$\text{therefore BE}\times\text{ST}=\frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2\times 4bx}}{b}$$

Now the area of any ellipsis being in a constant ratio to the rectangle of its greater and lesser axes (namely, as 3,14159, &c. to 4) the last general expression must therefore be a *maximum*, when the area is so; and therefore its fluxion, or that of $b^2x\times\overline{b+x}|^2+a^2x\times\overline{b-x}|^2$ $(=b^4x+2b^3x^2+b^2x^3+a^2b^2x$
* Art. 22. $-2a^2bx^2+a^2x^3)$ equal to nothing;* that is, $b^4\dot{x}+4b^3x\dot{x}+3b^2x^2\dot{x}+a^2b^2\dot{x}-4a^2bx\dot{x}+3a^2x^2\dot{x}=0$:

whence $x^2-\frac{4bx\times\overline{a^2-b^2}}{3a^2+3b^2}=-\frac{b^2}{3}$, and $x=$

$\frac{2b\times\overline{a^2-b^2}\pm b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$; from which the ellipsis is known.

But it is observable, that, when $a^4-14a^2b^2+b^4$ is negative, this solution fails, because the square root of a negative quantity is to be extracted. Therefore, to determine the limit, put $a^4-14a^2b^2+b^4=0$; then, by ordering the equation, you will get $a^2=b^2\times\overline{7+\sqrt{48}}$, and $a=b\times\overline{2+\sqrt{3}}$; and therefore $a:b::2+\sqrt{3}:1$. Hence, if the ratio of A C to C D be not greater than that of $2+\sqrt{3}$ to 1, or (which comes to the same thing) if the angle DAC be not less than 15 degrees, the fluxion of the ellipsis can never become equal to nothing; but the ellipsis itself will increase continually from the vertex till it coincides with the base of the cone; and therefore is greater at the base than in any other position.

But it is further to be observed, that this problem is confined to yet narrower limits. For either the ellipsis will increase, continually, from the vertex to the base of the cone (which is shown to be the case when the angle D A C is greater than 15°) or else it will increase till the point E arrives at a certain position H, and afterwards decrease to another certain position *h*, and then increase again till it coincides with the base of the cone (for it must always increase again before it coincides with the base, because, after the point E is got below the perpendicular B Q, both the axes of the ellipsis increase at the same time).

The same thing also appears from the foregoing equation $x=\dfrac{2b\times\overline{a^2-b^2}\pm b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$; whose two roots express the two values of $x$ (or C $v$) at the times of the *maximum* (at H) and its succeeding *minimum* (at *h*). Hence it is manifest, that the ellipsis may admit of a *maximum* between the vertex of the cone and the perpendicular B Q, and yet that *maximum* be less than the base of the cone, unless the foresaid angle D A C be so much less than 15° (above found) that the increase from *h* to D be less than the decrease from H to *h*. Now, therefore, to determine the exact limit, let the foresaid increment and decrement be supposed equal to each other, or, which is the same in effect, let the ellipsis B T E S B = the circle B $q$ D $m$, or B E × S T = B D², that is, let

$$\frac{\sqrt{\overline{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2}\times\overline{4bx}}}{b}=4b^2:$$ from which

equation you will get $a^2 = \frac{b^2}{x} \times \frac{4b^3 - b^2x - 2bx^2 - x^3}{\overline{b-x}^2}$ $= \frac{b^2}{x} \times \frac{4b^2 + 3bx + x^2}{b-x}$: moreover, from the equation $b^4\dot{x} + 4b^3x\dot{x} + 3b^2x^2\dot{x} + a^2b^2\dot{x} - 4a^2bx\dot{x} + 3a^2x^2\dot{x} = 0$, (given above), you will again get $a^2 = \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{-b^2 + 4bx - 3x^2}$ $= \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{\overline{b-x} \times \overline{3x-b}}$: whence, by comparing these equal values, there arises $\frac{4b^2 + 3bx + x^2}{x} = \frac{b^2 + 4bx + 3x^2}{3x-b}$ which, ordered, gives $x^2 + 2bx - b^2 = 0$, and therefore $x = b\sqrt{2} - b$.

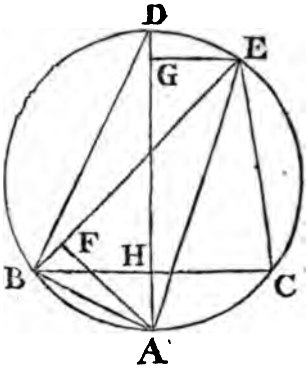
Moreover, $\frac{a^2}{b^2}$ being $= \frac{4b^2 + 3bx + x^2}{bx - x^2}$, if $b^2 - 2bx$ be substituted herein for its equal $x^2$, it will become $\frac{a^2}{b^2} = \frac{5b^2 + bx}{bx - x^2} = \frac{5b + x}{3x - b} = \frac{5b + b\sqrt{2} - b}{3b\sqrt{2} - 3b - b} = \frac{4 + \sqrt{2}}{-4 + 3\sqrt{2}}$ $= \frac{\overline{4 + \sqrt{2}} \times \overline{4 + 3\sqrt{2}}}{\overline{-4 + 3\sqrt{2}} \times \overline{4 + 3\sqrt{2}}} = \frac{22 + 16\sqrt{2}}{2} = 11 + 8\sqrt{2}$

Hence we have, $1 : \sqrt{11 + 8\sqrt{2}} :: b$ (DC) : $a$ (AC) :: radius to the tangent of the angle ADC = 78° 3′: whose complement DAC = 11° 57′, is the least limit possible. Therefore, unless the angle which the slant side makes with the axis be less than 11° 57′, the greatest ellipsis will be less than the base of the cone.

## EXAMPLE XVIII.

42. *Of all Triangles, having the same given Perimeter, and inscribed in the same given Circle; to determine the greatest.*

Let the diameter DA bisect the base BC of the required triangle BEC in H, draw AE, AB, and BD; also draw AF perpendicular to BE, and GE parallel to



BC, meeting AD in G: then, putting AD $= a$, half the given perimeter of the triangle $= b$, and DH$=y$; we have BH$=\sqrt{ay-y^2}$, and therefore EF$=b-\sqrt{ay-y^2}$.* Moreover DH $(y)$ : AD $(a)$ :: DB$^2$ : DA$^2$ :: EF$^2$ $\left(\overline{b-\sqrt{ay-y^2}}\right|^2)$ : EA$^2$ $= \frac{a}{y} \times \overline{b-\sqrt{ay-y^2}}\big|^2$; therefore AG $\left(\frac{\text{AE}^2}{\text{AD}}\right) = \frac{\overline{b-\sqrt{ay-y^2}}\big|^2}{y}$, and HG $=$ (AG$-$AH$=$) $\frac{b^2-2b\sqrt{ay-y^2}}{y}$; whence the area of the triangle BEC (BH$\times$HG) $= \frac{b^2\sqrt{ay-y^2}}{y} - 2ba + 2by$, whose fluxion $2b\dot{y} - \frac{\frac{1}{2}ab^2\dot{y}}{y\sqrt{ay-yy}}$ being put $= 0$, gives $y\sqrt{ay-yy} = \frac{1}{4}ba$; whence $y$, and from thence the sides of the triangle may be determined.

## EXAMPLE XIX.

43. *To determine the greatest Area that can be contained under four given Right-lines.*

Though it is demonstrable from common geometry that the area will be a *maximum* when the trapezium ABCD, formed by the given lines, may be inscribed in a circle,† yet I shall here give the solution from the principles of fluxions (whose uses I am now

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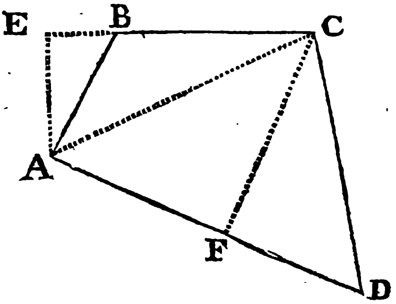
* By Prop. 13, page 62, *Elem. Trig.*
† See page 117, of *Elem. Geometry.*



illustrating). In order to which, let the diagonal A C be drawn, and upon C B and A D let fall the perpendiculars A E and C F; putting AB$=a$, BC$=b$, CD$=c$, DA$=d$, BE$=x$, and D F $=y$; then A E being $=\sqrt{a^2-x^2}$, and CF$=\sqrt{c^2-y^2}$, the area of the trapezium ($\frac{1}{2}$BC × AE + $\frac{1}{2}$AD × CF) will be$=\frac{1}{2}b\sqrt{a^2-x^2}+\frac{1}{2}d\sqrt{c^2-y^2}$; and its fluxion $\frac{-\frac{1}{2}bx\dot{x}}{\sqrt{a^2-x^2}}-\frac{\frac{1}{2}dy\dot{y}}{\sqrt{c^2-y^2}}=0$;* and therefore $\frac{-dy\dot{y}}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$. Moreover, since $b^2+a^2+2bx$ ($=$AC$^2$) $=d^2+c^2-2dy$, by taking the fluxion thereof, we have $2b\dot{x}=-2d\dot{y}$, or $-d\dot{y}=b\dot{x}$; which, substituted for $-d\dot{y}$ in the foregoing equation, gives $\frac{b\dot{x}y}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$, and $\frac{y}{\sqrt{c^2-y^2}}=\frac{x}{\sqrt{a^2-x^2}}$; and consequently, $\sqrt{c^2-y^2}$ (C F) : $y$ (D F) :: $\sqrt{a^2-x^2}$ (A E) : $x$ (B E): from which it appears that the triangles DCF and ABE are similar, and that (D+ABC being=2 right angles) the trapezium may be inscribed in a circle; but this by the bye. We are now to get an expression for the area in known terms, and in order thereto we have $b^2+a^2+2bx=d^2+c^2-2dy$, $y=\frac{cx}{a}$, and CF$=\frac{c\sqrt{a^2-x^2}}{a}$ (because AB : BE :: DC : DF, &c.): therefore, by substitution, $b^2+a^2+2bx=d^2+c^2-\frac{2cdx}{a}$, and the area ($\frac{1}{2}$ BC × A E

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* Art. 22.

$+\frac{1}{2}\mathrm{AD}\times\mathrm{CF}) = \frac{1}{2}b\sqrt{a^2-x^2} + \frac{cd}{2a}\cdot\sqrt{a^2-x^2} = \frac{ab+cd}{2a}\sqrt{a^2-x^2}$; and therefore the square thereof $= \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a^2-x^2} = \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a+x}\times\overline{a-x} = \frac{\overline{ab+cd}|^2}{4}\times\overline{1+\frac{x}{a}}\times\overline{1-\frac{x}{a}}$. But since $b^2+a^2+2bx = d^2+c^2-\frac{2cdx}{a}$, we have $\frac{x}{a} = \frac{d^2+c^2-b^2-a^2}{2ab+2cd}$, $1+\frac{x}{a} = 1+\frac{d^2+e^2-b^2-a^2}{2ab+2cd} = \frac{2ab+2cd+d^2+c^2-b^2-a^2}{2ab+2cd} = \frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}$; and $1-\frac{x}{a} = \frac{2ab+2cd-d^2-c^2+b^2+a^2}{2ab+2cd} = \frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd}$; and consequently the square of the area $= \frac{\overline{ab+cd}|^2}{4}\times\frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}\times\frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd} = \frac{\overline{\overline{d+c}|^2-\overline{b-a}|^2}\times\overline{\overline{b+a}|^2-\overline{d-c}|^2}}{16}$ which (because the difference of the squares of any two quantities is equal to a rectangle under their sum and difference) will also be $= \frac{\overline{d+c+b-a}\times\overline{d+c-b+a}\times\overline{b+a+d-c}}{4}\times\frac{\overline{b+a-d+c}}{4} = \overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-a}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-b}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-c}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-d}$. Whence it appears, that if from $\frac{1}{2}$ the sum of all the four sides each particular side be subtracted, the continual product of the remainders will be the square, or second power, of the area.

From this theorem, the rule in common practice, for finding the area of a triangle, having the three sides given, is deduced as a corollary: for, making

$a=0$. the trapezium becomes a triangle, and the second power of its area $= \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b} \times \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-b} \times \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-c} \times \overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-d}$: which, in words, is the common rule.

## EXAMPLE XX.

44. *To find the greatest Value of y in the Equation* $a^4x^2 = \overline{x^2+y^2}|^3$.

By putting the whole equation into fluxions, &c. we have $2a^4x\dot{x} = \overline{2x\dot{x}+2y\dot{y}} \times 3 \times \overline{x^2+y^2}|^2$; which in the required circumstance, when $\dot{y}=0$,* becomes $2a^4x\dot{x} = 6x\dot{x} \times \overline{x^2+y^2}|^2$; whence $x^2+y^2 = \frac{a^2}{\sqrt{3}}$, and $\overline{x^2+y^2}|^3 = \frac{a^6}{3\sqrt{3}}$: but, by the given equation $\overline{x^2+y^2}|^3 = a^4x^2$; consequently $a^4x^2 = \frac{a^6}{3\sqrt{3}}$, and therefore $x = a\sqrt{\frac{1}{3\sqrt{3}}}$; whence $y^2 \; (=\frac{a^2}{\sqrt{3}} - x^2) = \frac{2a^2}{3\sqrt{3}}$, and $y = a\sqrt{\frac{2}{3\sqrt{3}}}$.

* Art. 22.

### *The same otherwise.*

Since $\overline{x x+yy}|^3$ is given $= a^4x^2$, we have $x^2+y^2 = a^{\frac{4}{3}} \times x^{\frac{2}{3}}$, and therefore $y^2 = a^{\frac{4}{3}} \times x^{\frac{2}{3}} - x^2$; whose fluxion, $\frac{2}{3}a^{\frac{4}{3}} \times x^{-\frac{1}{3}}\dot{x} - 2x\dot{x}$, being put $=0$, we also get $\frac{a^{\frac{4}{3}} \times x^{-\frac{1}{3}}}{3} = x$; whose cube is $\frac{a^4 \times x^{-1}}{27} = x^3$, or $\frac{a^4}{27x} = x^3$; whence $27x^4 = a^4$, and consequently $x = a\sqrt{\frac{1}{3\sqrt{3}}}$, the same as before.

45. When, in the general expression, whose *maximum* or *minimum* is sought, there are two or more indeterminate quantities, independent of each other, their respective values, in the required circumstance, will be determined, by making them flow, one by one, while the others are supposed invariable; as in the following

## EXAMPLE XXI.

*Wherein it is proposed to find three such Values of $x$, $y$, and $z$, as shall make the Value of $\overline{b^3-x^3}\times\overline{x^2z-z^3}\times\overline{xy-y^2}$ the greatest possible.*

First, considering $y$ as variable, and the rest constant, we have $x\dot{y}-2y\dot{y}=0$;\* whence $y=\frac{1}{2}x$, and $xy-y^2=\frac{1}{4}x^2$. By making $z$ variable, we have $x^2\dot{z}-3z^2\dot{z}=0$; whence $z=\frac{x}{\sqrt{3}}$, and $x^2z-z^3=\frac{2x^3}{3\sqrt{3}}$. Now let these values of $xy-y^2$ and $x^2z-z^3$ be substituted in the given expression, and it will become $\frac{x^2}{4}\times\frac{2x^3}{3\sqrt{3}}\times\overline{b^3-x^3}=\frac{b^3x^5-x^8}{6\sqrt{3}}$; therefore $5b^3x^4\dot{x}-8x^7\dot{x}=0$: whence $x=\frac{1}{2}b\times\sqrt[3]{5}$, $y\ (=\frac{1}{2}x)=\frac{1}{4}b\times\sqrt[3]{5}$, and $z\ \left(=\frac{x}{\sqrt{3}}\right)=\frac{1}{2}b\times\frac{\sqrt[3]{5}}{\sqrt{3}}$.

The reason of the foregoing process is obvious: for, if the fluxion of the given expression, when any one of the indeterminate quantities is made variable, be not equal to nothing, that expression may become greater, without altering the values of the rest, which are considered as constant:† and therefore cannot be the greatest possible, unless the said fluxion is equal to nothing.

* Art. 22.

† Art. 22.

## EXAMPLE XXII.

46. *To determine the different values of $x$, when that of $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$ becomes a* Maximum *or* Minimum.

The fluxion of the given Expression being *(as usual)* put equal to nothing, we have $12x^3-84ax^2+168a^2x-96a^3=0$, or $x^3-7ax^2+14a^2x-8a^3=0$: from whence (by the method of divisors) we get $x-a=0$, $x-2a=0$, or $x-4a=0$: therefore, the roots of the equation, or the three values of $x$, are $a$, $2a$, and $4a$.

### Scholium.

47. It appears, from the last example, that a quantity may admit of as many *maxima* and *minima* (according to the meaning of the definition*) as there are possible roots in the equation, arising from assuming its fluxion equal to nothing. Now to know which of those roots point out a *maximum*, and which a *minimum*; find whether the value of the said fluxion, a little before it becomes equal to nothing, be positive or negative; if *positive*, the succeeding root gives a *maximum*; but if *negative*, a *minimum*: the reason of which is extremely obvious; because so long as any quantity increases, its fluxion is positive, but when it decreases, the fluxion is negative.

As an example hereof, let the quantity $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$, be again resumed; whose fluxion is $12\dot{x}\times\overline{x^3-7ax^2+14a^2x-8a^3}=12\dot{x}\times\overline{x-a}\times\overline{x-2a}\times\overline{x-3a}$: whereof the value, before it becomes equal to nothing, the first time (or before $x=a$) being negative (because the product of three negative factors is negative) its first root $(a)$ therefore indicates a *minimum*: whence we may conclude, without considering farther, that the second root $(2a)$ gives a *maximum*, and the third $(4a)$ another *minimum*. But, if you would know whether the first or third root gives the lesser value of the two; it is but substituting in the given quantity, which will come out $48b^4-37a^4$, and $48b^4-64a^4$ respectively; therefore the latter is the lesser, and the very least value the proposed expression can admit of.

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* Art. 22.

When all the roots prove impossible, the quantity proposed (as its fluxion can never become $=0$) must either increase, or decrease, continually; and therefore can neither admit of a *maximum* nor a *minimum.*

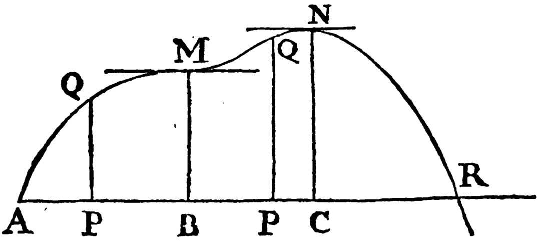
Moreover, it may so happen, that the roots are possible, the fluxion $=0$, and yet the quantity itself be neither a *maximum* nor a *minimum* in that circumstance.

For let us again suppose the point *n* to move after *m*, as in the general illustration *(vid. Art.* 22), only let the velocity of *n (in the first case)* increase no longer than 'till it arrives at D; after which let it again decrease: then, though the fluxion of the distance *mn*, is nothing, at the position C D, yet the distance itself will not be a *maximum;* because *n* (having afterwards, as well as before, a less velocity than *m*) will still continue to lose ground.—In the same manner the matter may be explained with regard to a *minimum.* And it is evident, that these cases will always happen when the fluxion of the given quantity is of the same denomination (with regard to positive and negative) both before and after, it becomes equal to nothing: which, by the rules of common algebra, is known to be when the equation admits of an even number of equal roots.—An example hereof, however, may not be improper.

Let then the quantity proposed be $24a^3x-30a^2x^2+16ax^3-3x^4$; whose fluxion is $24a^3\dot{x}-60a^2x\dot{x}+48ax^2\dot{x}-12x^3\dot{x}=12\dot{x}\times\overline{a-x}\times\overline{a-x}\times\overline{2a-x}$: which being made $=0$, it appears that the two least roots are equal. Therefore there is neither a *maximum* nor *minimum* when $x=a$ (because whether $x$ be taken a little less, or a little greater, than $a$, the value of the fluxion will still be affirmative). The greatest root, however, not being affected with another equal one, indicates a *maximum*, according to the rule above prescribed.

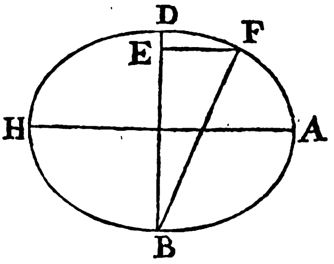
To render what has been observed above still more conspicuous, let the given expression, $24a^3x-30a^2x^2+16ax^3-3x^4$, be represented by the variable ordinate P Q of the curve AQMNR, whose abscissa AP is (as usual) denoted by $x$.

Then, whilst ($12\dot{x}\times\overline{a-x}\times\overline{a-x}\times\overline{2a-x}$) the fluxion of the ordinate continues positive (or till $x$ becomes $=a=$ AB), the ordinate itself will increase: but at the position B M it becomes stationary (if I may be allowed the expression) the fluxion being then $=0$. After which, the fluxion being again affirmative, the ordinate will again increase, till $x$ becomes $=2a$ ($=$ A C); when, the fluxion becoming nothing (a second time), and afterwards negative, CN will be a *maximum:* soon after which the curve descends below its axis, and continues to recede from it *in infinitum.*



Another thing there is that ought to be regarded in the solution of these kinds of problems, and that is, whether the *maxima* or *minima*, found by assuming the fluxion $=0$, fall within the limits prescribed by the nature of the question or figure; which is often restrained by conditions that do not enter into the algebraic computation.

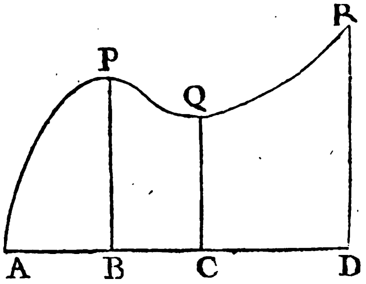
Thus, for example; suppose it were required to find that point (F) in a given ellipsis ABHD which, of all others, is the most remote from the extreme B of the conjugate axis BD.



Then, drawing FE parallel to the transverse A H, and putting $AH=a$, $BD=b$, and $BE=x$, we have, by the property of the curve $BF^2$ $(=BE^2+EF^2)$ $x^2+\overline{bx-x^2}\times\frac{a^2}{b^2}$; from whence $x$ is found $=\frac{\frac{1}{2}a^2b}{a^2-b^2}$. But, from the nature of the figure, the greatest value that $x$ $(=BE)$ can possibly admit of is $b$ $(=BD)$, therefore if the relation of $a$ and $b$ be such, that $\frac{\frac{1}{2}a^2b}{a^2-b^2}$ is greater than $b$, this solution is manifestly impossible.—— To determine the limit, therefore, make $\frac{\frac{1}{2}a^2b}{a^2-b^2}=b$; then it will be found that $2b^2=a^2$. Whence the foregoing solution can only obtain when $2BD^2$ is equal to, or less than $AH^2$.

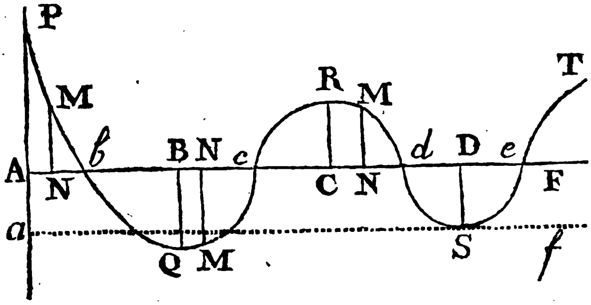
Again, it ought to be also considered whether the value of $x$, found by the common method, gives a less quantity for the *maximum*, and a greater for the *minimum*, than will arise from the extremes themselves by which $x$ is limited.

Thus, let it be required to determine the greatest and least ordinates in a curve, APR, whose equation is $y^3=6a^2x-9ax^2+4x^3$, and whose greatest abscissa AD is given equal $2a$.



Here we shall, by taking the fluxion, *&c.* have $x = \frac{1}{2}a$, or $x = a$: the former of which values gives the corresponding ordinate $BP = a\sqrt[3]{\frac{5}{4}}$; and the latter, $CQ = a$: but the first of these is not the greatest of all others, because the extreme DR exceeds it, being $= 2a$; nor is CQ the least possible, because the ordinate at the other extreme A is nothing at all.

Sometimes one, or more, of the points Q, S, *&c.* determining the *maxima* and *minima,* will fall below the axis AF *(as in the annexed figure).* In which case the corresponding value of the general expression, represented by the ordinate, will be negative: but at the points *b*, *c*, *d*, *&c.* where the curve intersects the axis, it will be equal to nothing: whence (by-the bye) the reason why the roots of an equation ($x^n - ax^{n-1} + b^2x^{n-2} \ldots + q^n = 0$) are impossible by pairs is evident. For, seeing A*b*, A*c*, A*d*, A*e*, *&c.* are the roots of that equation, or the different values of $x$, when the ordinate $x^n - ax^{n-1} + b^2x^{n-2} \ldots\ldots + q^n$ (MN) becomes equal to nothing, it is plain, if PA, expressing the given term $q^n$, be increased to P*a*, so that AF (then coinciding with *af*) may touch the curve in S, the adjacent roots A*d* and A*e* will then become



equal; and if $q^n$ be farther increased, so that the axis may fall wholly below the curve, not only those two, but also the other roots, A$b$ and A$c$, will become impossible.

Various other observations might be made, relating to the limits of equations, determined by these *maxima* and *minima*; but this being foreign to the matter in hand, I shall content myself with one remark more, *viz.*

*Any expression, which being put equal to nothing, admits of two or more equal roots, has as many succeeding orders of fluxions equal to nothing, at the same time, as are expressed by the number of those roots minus one.*

Thus, an equation, having three equal roots, has both its first and second fluxions equal to nothing, when the fluent itself is equal to nothing.

Hence we have another way (besides that given above) to know when a quantity may have its fluxion equal to nothing, and yet neither admit of a *maximum* nor a *minimum*: for, since this circumstance always takes place when the equation admits of an *even* number of equal roots (as has been already shown) the number of orders of fluxions, equal to nothing, at the same time (including the first) must also be even.

Hence, also, we have an easy method for discovering when some of the roots of an equation are equal; and, if so, what they are.

Thus, let $x^3 - 3ax^2 + 4a^3 = 0$ be propounded; whereof the fluxion $3x^2\dot{x} - 6ax\dot{x}$ being assumed equal to nothing, we find $x = 2a$; which will also be a root of the given equation, if it admits of two equal ones: to try it, therefore, I substitute $2a$ for $x$, and find it answers.

Again, let $8x^4 - 28ax^3 + 18a^2x^2 + 27a^3x - 27a^4 = 0$; whereof the first and second fluxions being $32x^3\dot{x} - 84ax^2\dot{x} + 36a^2x\dot{x} + 27a^3\dot{x}$ and $96x^2\dot{x}^2 - 168ax\dot{x}^2 + 36a^2\dot{x}^2$, if the latter of them be assumed $= 0$, $x$ will be found$=\frac{7a}{8}\pm\sqrt{\frac{25a^2}{64}}=\frac{3a}{2}$, or $\frac{a}{4}$: one of which quantities, if the equation proposed admits of three equal roots, will be the value of each of them: by trying $\frac{3a}{2}$, it will be found to succeed. Whence, by a well known rule, the fourth root (being$=\frac{28a}{8}-\frac{3a}{2}\times 3=-a$) is also given.

The reason of these operations, as well as what is asserted above, may be thus demonstrated.

Let $\overline{r-x}\times\overline{r-x}$ &c. $\times\ \overline{A+Bx+Cx^2\ \&c.}=0$, be any equation, having two or more equal roots, represented, each, by $r$: put $y=r-x$, and let the number of the equal roots be denoted by $n$; then, by substitution, we have $y^n\times A+B\times\overline{r-y}+C\times\overline{r-y}|^2$ &c. $=0$; which, by expanding the powers of $r-y$, and putting $a=A+Br+Cr^2$ &c. $b=B+2Cr+3Dr^2$ &c. will be further transformed to $y^n\times\overline{a-by+cy^2-dy^3\ \&c.}=0$: whose fluxion $na\dot{y}y^{n-1}-\overline{n+1}\,.\,b\dot{y}y^n+\overline{n+2}\times c\dot{y}y^{n+1}$ &c. is evidently equal to nothing, when $y$, or its equal $r-x$, is nothing (provided $n$ be greater than unity). It is equally plain, that the second fluxion $n\,.\,\overline{n-1}\,.\,a\dot{y}^2y^{n-2}-\overline{n+1}\,.\,nb\dot{y}^2y^{n-1}+\overline{n+2}\,.\,\overline{n+1}\,.\,c\dot{y}^2y^n$ &c. will also be equal to nothing, in the same circumstance, if $n$ be greater than 2, &c. &c.

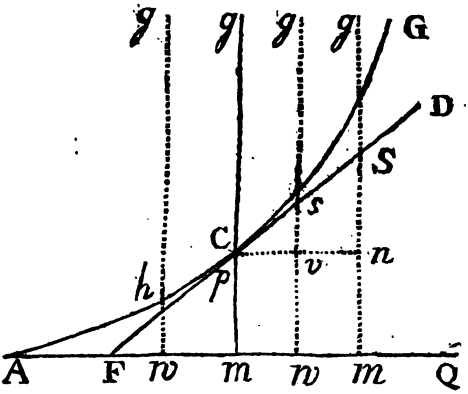
Hence, universally, let the number $(n)$ of equal roots be what it will, that of the orders of fluxions equal to nothing, at the same time, will be expressed by that number *minus* one, as was to be shown.

## SECTION III.

### *The Use of* FLUXIONS *in drawing Tangents to Curves.*

### ILLUSTRATION.

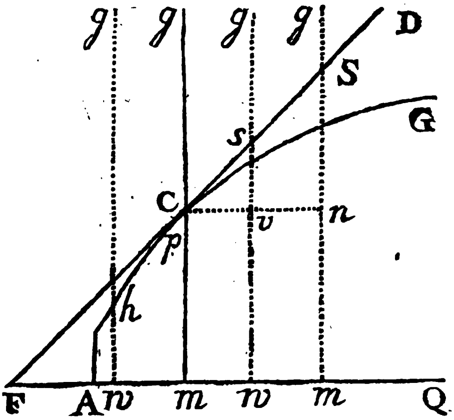
48. LET A C G be a curve of any kind, and C the given point from whence the tangent is to be drawn.



Conceive a right-line *mg* to be carried along uniformly, parallel to itself, from A towards Q, and let, at the same time, a point *p* so move in that line, as to describe, or trace out, the given curve A C G: also let *mm*, or C*n* (equal and parallel to *mm*) express the fluxion of A *m*, or the celerity wherewith the line *mg* is carried; and let *n* S express the corresponding fluxion of *mp*, in the position *m* C *g*, or the celerity of the point *p*, in the line *mg*. Moreover, through the point C let the right-line S F be drawn, meeting the axis of the curve (A Q) in F.

Now, it is evident, if the motion of $p$, along the line $mg$, was to become equable at $C$, the point $p$ would be at $S$, when the line itself had acquired the position $mSg$ (because, by hypothesis, $Cn$ and $nS$ express the distances that might be described by the two uniform motions in the same time).

And, if $wsg$ be assumed to represent any other position of that line, and $s$ the contemporary position of the point $p$ (still supposing an equable celerity of $p$); then the distances $Cv$ and $vs$, gone over, in the same



time, by the two motions, will, always, be to each other as the celerities, or as $Cn$ to $nS$: therefore, since $Cv : vs :: Cn : nS$ (which is a known property of similar triangles) the point $s$ will, always, fall in the right-line $FCS$: whence it appears, that, if the motion of the point $p$ along the line $mg$ was to become uniform at $C$, that point would then move in the right-line $CS$, instead of the curve-line $CG$.

Now, seeing the motion of $p$, in the description of curves, must, either, be an accelerated or a retarded one, let it be, first, considered as an accelerated one: in which case the arch $CG$ will fall, *wholly*, above the right-line $CD$ (as in fig. 1), because the distance of the point $p$ from the axis AQ, at the end of any given time, is greater than it would be if the acceleration was to cease at C; and, if the acceleration had ceased at C, the point $p$ would (it is proved) have been always found in the said right-line F S.

But if the motion of the point $p$ be a retarded one, it will appear, by reasoning in the same manner, that the arch C G will fall wholly below the right-line C D (as in fig. 2).

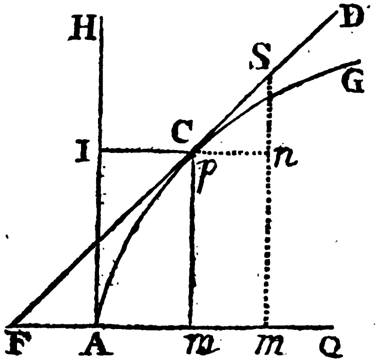
This being the case, let the line $mg$, and the point $p$, along that line, be now supposed to move back again, towards A and $m$, in the same manner they proceeded from thence: then, since the celerity of $p$ (*fig.* 1) did before increase, it must now, on the contrary, decrease; and, therefore, as $p$, at the end of a given time, after repassing the point C, is not so near to AQ, as it would have been, had the velocity continued the same as at C, the arch C$h$ (as well as C G) must fall wholly above the right-line F C D. And, by the same method of arguing, the arch C$h$, *in the second case*, will fall, *wholly*, below F C D: therefore F C D, *in both cases*, is a tangent to the curve at the point C. whence, the triangles F$m$C and C$n$S being similar, it appears, that the sub-tangent $m$F is always a fourth-proportional to ($n$S) the fluxion of the ordinates (C$n$), the fluxion of the abscissa, and the ordinate (C$m$).

*Otherwise.*

49. Let ACG represent the proposed curve, and let the right-line F C D be a tangent to it, at any point C, meeting the axis AQ (produced if necessary) in F: suppose a point $p$ to move along the curve, from A towards G, and let the absolute celerity thereof at C, in the direction of the tangent C D, or the fluxion of the line A$p$ so generated*, be denoted by C S, any part of the said tangent: then, if A H, $mp$ and $m$S be made perpendicular, and I$pn$ parallel, to AQ, the relative celerities of that point, in the directions C$n$ and $m$C, wherewith I$p$ (=A$m$) and $mp$ increase in this

*Art. 2&5.

position, will be truly expressed by C $n$ and $n$ S*: but the celerities by which quantities increase are as the fluxions of those quantities: therefore (C S being the fluxion of the curve line A$p$) C $n$ and $n$ S are the corresponding fluxions of the abscissa A $m$ and the ordinate $m\,p$: and we have S$n$ : $n$C :: $m$ C : $m$F, *the same as before.*

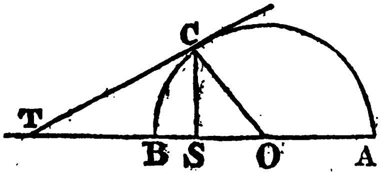


Hence, if the abscissa A $m$ be put $=x$, and the ordinate $m\,p=y$, we shall have $mF=\frac{y\dot{x}}{\dot{y}}$: by means of which general expression, and the equation expressing the relation between $x$ and $y$, the ratio of the fluxions $\dot{x}$ and $\dot{y}$ will be found, and from thence the length of the sub-tangent ($m$ F) as in the following Examples:

## EXAMPLE I.

50. *To draw a Right-line* C T, *to touch a given Circle* B C A, *in a given Point* C.

Let C S be perpendicular to the diameter A B, and put A B $=a$, B S $=x$ and S C $=y$: then, by the property of the circle, $y^2$ (C S$^2$) $=$ B S $\times$ A S($=x\times\overline{a-x}$) $=ax-x^2$;



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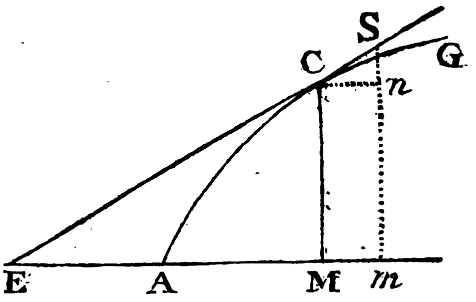
[*] Art. 35.

whereof the fluxion being taken, in order to determine the ratio of $\dot{x}$ and $\dot{y}$, we get $2y\dot{y}=a\dot{x}-2x\dot{x}$; consequently $\frac{\dot{x}}{\dot{y}}=\frac{2y}{a-2x}=\frac{y}{\frac{1}{2}a-x}$; which, multiplied by $y$, gives $\frac{y\dot{x}}{\dot{y}}=\frac{y^2}{\frac{1}{2}a-x}=$ the sub-tangent S T*. Whence (O being supposed the center) we have OS ($\frac{1}{2}a-x$) : C S ($y$) : : C S ($y$) : S T; which we also know from other principles.

## EXAMPLE II.

### 51. *To draw a Tangent to any given Point* C *of the conical Parabola* A C G.

If the *Latus Rectum* of the curve be denoted by $a$, the ordinate M C by $y$, and its corresponding abscissa



A M by $x$; then the known equation, expressing the relation of $x$ and $y$, being $ax=y^2$, we have, in this case, $a\dot{x}=2y\dot{y}$; whence $\frac{\dot{x}}{\dot{y}}=\frac{2y}{a}$, and consequently $\frac{y\dot{x}}{\dot{y}}$† $=\frac{2y^2}{a}=\frac{2ax}{a}=2x=$ M F. Therefore the sub-tangent is just the double of its corresponding abscissa A M: which we likewise know from other principles.

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* Art. 48 & 49.

† Art. 48 & 49.

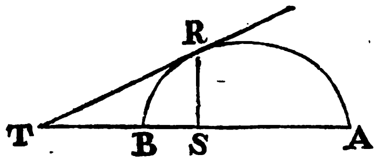
## EXAMPLE III.

### 52. *To draw a Tangent to a Parabola of any kind.*

The general equation of these sort of curves being $a^m x^n = y^{m+n}$, we have $na^m x^{n-1} \dot{x} = \overline{m+n} \times y^{m+n-1} \dot{y}$, and thererefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{m+n} \times y^{m+n-1}}{na^m x^{n-1}}$; whence $\frac{y\dot{x}}{\dot{y}} = \frac{\overline{m+n} \times y^{m+n}}{na^m x^{n-1}} = \frac{\overline{m+n} \times a^m x^n}{na^m x^{n-1}}$ (because $y^{m+n} = a^m x^n$) $= \frac{m+n}{n} \times x =$ the true value of the sub-tangent: which, therefore, is to the abscissa, in the constant ratio of $m+n$ to $n$.

## EXAMPLE IV.

### 53. *To draw a Tangent* R T, *to a given Point* R, *in a given Ellipsis* B R A.



If R S be an ordinate to the principal axis A B, and there be put *(as usual)* $BS = x$, $RS = y$, $AB = a$, and the lesser axis $= b$; we shall, by the property of the curve, have $a^2 : b^2 :: ax - x^2$ (BS × AS) $: y^2$ ($RS^2$), and therefore $b^2 \times \overline{ax - x^2} = a^2y^2$: whence $b^2 \times \overline{a\dot{x} - 2x\dot{x}} = 2a^2y\dot{y}$, and $\frac{\dot{x}}{\dot{y}} = \frac{2a^2y}{b^2 \times \overline{a - 2x}}$; and consequently the sub-tangent

* Art. 49. $$ST \left(\frac{y\dot{x}}{\dot{y}}\right)^{*} = \frac{2a^2y^2}{b^2 \times \overline{a - 2x}} = \frac{a^2y^2}{b^2 \times \overline{\tfrac{1}{2}a - x}} = \frac{b^2 \times \overline{ax - x^2}}{b^2 \times \overline{\tfrac{1}{2}a - x}} =$$

$\frac{ax-x^2}{\frac{1}{2}a-x}$. Whence the point T being given, through which the tangent must pass, the tangent itself may be drawn.

But if you would derive an expression for the sub-tangent, in any other kind of ellipsis (besides the conical) let the equation $\overline{a-x}|^m \times x^n = \frac{c}{a} \times y^{m+n}$, exhibiting the nature of all kinds of ellipses, be assumed: then, by taking the fluxion thereof, you will have $-m\dot{x} \times \overline{a-x}|^{m-1} \times x^n + n\dot{x}x^{n-1} \times \overline{a-x}|^m = \frac{c}{a} \times \overline{m+n} \times y^{m+n-1}\dot{y}$; and therefore $\frac{y\dot{x}}{\dot{y}} =$

$$\frac{\frac{c}{a} \times \overline{m+n} \times y^{m+n}}{-m \times \overline{a-x}|^{m-1} \times x^n + nx^{n-1} \times \overline{a-x}|^m}$$

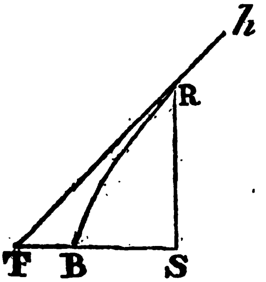
$= \frac{\overline{m+n} \times \overline{a-x}|^m \times x^n}{-mx^n \times \overline{a-x}|^{m-1} + nx^{n-1} \times \overline{a-x}|^m}$ (because $\frac{c}{a} \times y^{m+n} = \overline{a-x}|^m \times x^n$) $= \frac{\overline{m+n} \times \overline{a-x} \times x}{-mx+n \times \overline{a-x}} =$ $\frac{\overline{m+n} \times \overline{ax-x^2}}{na-\overline{n+m} \times x}$; which is the sub-tangent required.

## EXAMPLE V.

54. *To draw a Tangent to any given point R, in a given Hyperbola* B R *h*.

If $a$ and $c$ be put to denote the two principal diameters of the hyperbola, the equation of the curve will be $c^2 \times \overline{ax+x^2} = a^2y^2$: from whence we have $c^2 \times$

$\overline{a\dot{x}+2x\dot{x}}=2a^2y\dot{y}$, $\therefore \frac{\dot{x}}{\dot{y}}=\frac{a^2y}{c^2\times\overline{\frac{1}{2}a+x}}$, and consequently $\frac{y\dot{x}}{\dot{y}}=\frac{a^2y^2}{c^2\times\overline{\frac{1}{2}a+x}}$ $=\frac{c^2\times\overline{ax+x^2}}{c^2\times\overline{\frac{1}{2}a+x}}=$ $\frac{ax+x^2}{\frac{1}{2}a+x}=$ S T: whence B T (S T − B S) $=\frac{\frac{1}{2}ax}{\frac{1}{2}a+x}$ is also known; and therefore the point T being given, the tangent R T may be drawn.



The manner of drawing tangents to all sorts of hyperbolas, *universally*, will be the same as in the ellipses, the equations of the two kinds of curves differing in nothing but their signs.

## EXAMPLE VI.

55. *Let the proposed Curve be that whose Equation is*

$$ax^2+xy^2+x^3-y^3=0.$$

Then we shall have $2ax\dot{x}+y^2\dot{x}+2xy\dot{y}+3x^2\dot{x}-3y^2\dot{y}=0$; therefore $2ax\dot{x}+y^2\dot{x}+3x^2\dot{x}=3y^2\dot{y}-2xy\dot{y}$, $\frac{\dot{x}}{\dot{y}}=\frac{3y^2-2xy}{2ax+y^2+3x^2}$, and consequently $\frac{y\dot{x}}{\dot{y}}=\frac{3y^3-2xy^2}{2ax+y^2+3x^2}$ *.

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* Art. 48. & 49.

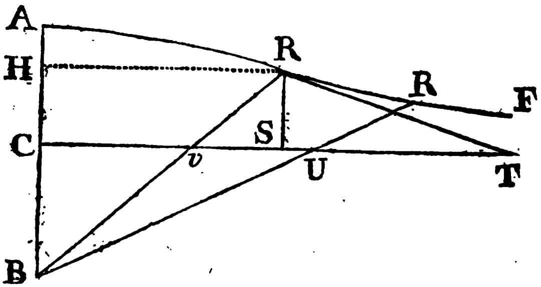
## EXAMPLE VII.

56. *Let the given Curve be the Cissoid of* Diocles, *whose Equation is* $y^2 = \frac{x^3}{a-x}$.

Here we have $2y\dot{y} = \frac{3x^2\dot{x} \times \overline{a-x} + \dot{x}x^3}{\overline{a-x}|^2} = \frac{3ax^2\dot{x} - 2x^3\dot{x}}{\overline{a-x}|^2}$; whence $\frac{\dot{x}}{\dot{y}} = \frac{2y \times \overline{a-x}|^2}{3ax^2 - 2x^3}$, and consequently the subtangent $\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2y^2 \times \overline{a-x}|^2}{3ax^2 - 2x^3} = \frac{2x^3}{a-x} \times \frac{\overline{a-x}|^2}{3ax^2 - 2x^3} = \frac{2x \times \overline{a-x}}{3a - 2x}$.

## EXAMPLE VIII.

57. *Let the Conchoid of* Nicomedes *be proposed;* whereof the nature is such, that, if from a point B, called the Pole, any number of right-lines, B A, B R, B R, *&c.* be drawn, the parts of those lines C A, *v* R, U R, *&c.* intercepted by the curve and its axis C T, shall be, all, equal to each other.

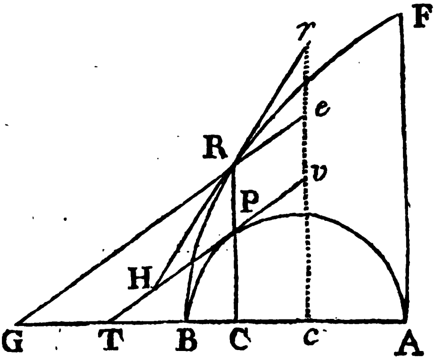


In this case (supposing A B and R S perpendicular, and R H parallel to C T; and putting B C$=a$, R $v$ (A C)$=b$, C S$=x$, and R S$=y$) we have, *per sim. Triang.* $a+y$ (BH) : $x$ (RH) :: $y$ (RS) : $\frac{xy}{a+y}=$ S$v$: but S$v$ ($\sqrt{\text{R}v^2-\text{RS}^2}$) is also$=\sqrt{b^2-y^2}$; therefore $\frac{xy}{a+y}=\sqrt{b^2-y^2}$, or $x^2y^2=\overline{a+y}|^2\times\overline{b^2-y^2}$ is the general equation of the curve; which, in fluxions, gives $2x^2y\dot{y}+2y^2x\dot{x}=2\dot{y}\times\overline{a+y}\times\overline{b^2-y^2}-2y\dot{y}\times\overline{a+y}|^2$ $=2\dot{y}\times\overline{a+y}\times\overline{b^2-ay-2y^2}$; and therefore $\frac{\dot{x}}{\dot{y}}=$ $\frac{\overline{a+y}\times\overline{b^2-ay-2y^2}-x^2y}{xy^2}$, consequently $\frac{y\dot{x}}{\dot{y}}=$ $\frac{\overline{a+y}\times y\times\overline{b^2-ay-2y^2}-x^2y^2}{y\times xy}=$ $\frac{\overline{a+y}\times y\times\overline{b^2-ay-2y^2}-\overline{a+y}|^2\times\overline{b^2-y^2}}{y\times\overline{a+y}\times\sqrt{b^2-y^2}}$ (because $x^2y^2$ $=\overline{a+y}|^2\times\overline{b^2-y^2}$)$=\frac{b^2y-ay^2-2y^3-ab^2+ay^2-b^2y+y^3}{y\sqrt{b^2-y^2}}$ $=\frac{-ab^2-y^3}{y\sqrt{b^2-y^2}}$: which being a negative quantity, the tangent will therefore fall on the contrary side of the ordinate, from the vertex; and so, by changing the signs we shall have $\frac{ab^2+y^3}{y\sqrt{b^2-y^2}}$ for the sub-tangent S T in this case.

After the manner of these examples the sub-tangent, in curves whose abscissas are right-lines, may be determined: but if the abscissa, or line terminating the ordinate, on the lower part, be another curve, then the tangent may be drawn as in the following:

## EXAMPLE IX.

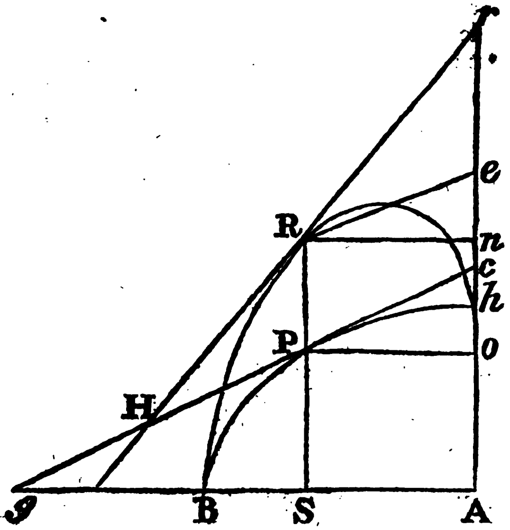
58. Let the curve B R F be a cycloid; whose abscissa is here supposed to be the semicircle B P A, to which let the tangent PT be drawn (as above). Moreover let $r$ R H be a tangent to the cycloid, at the corresponding point R, and let G R $e$ be parallel to TP$v$; putting the arch (or abscissa) B P $= z$, its ordinate PR$=y$, AF$=b$, and BPA$=c$: then by the property of the curve, we shall have $c$ (B PA) : $b$ (A F) :: $z$ (BP) : $y$ (PR) : therefore $y=\frac{bz}{c}$, and $\dot{y}=\frac{b\dot{z}}{c}=re$: but, by similar triangles, $re$ ($\dot{y}$) : R$e$ (=P $v=\dot{z}$) :: P R ($y$) : P H$=\frac{y\dot{z}}{\dot{y}}=z$ (because $y=\frac{bz}{c}$). Therefore, if in the right-line PT, there be taken P H equal to the arch P B, you will have a point H, through which the tangent of the cycloid must pass.



## EXAMPLE X.

59. Let B P $h$ be a curve of any kind, to which the method of drawing the tangent $c$ P $g$ is known; let

B R $h$ be another curve of such a nature, that the ordinate P R $(y)$ shall always be a mean proportional be-



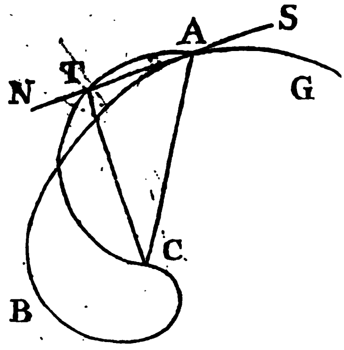
tween BS $(x)$ and AS $(a-x)$ supposing RPS perpendicular Art. 48 & 49. to A B: put P $o = \dot{x}$, S P $= v$, $o\,c = \dot{v}$*, and $e\,r = \dot{y}$: then *(as above)*, $e\,r$ $(\dot{y})$ : R $e$ $(= \text{P}\,c = \sqrt{\dot{x}^2+\dot{v}^2})$ :: R P $(y)$ : P H $= \frac{y\sqrt{\dot{x}^2+\dot{v}^2}}{\dot{y}}$: but, by the equation of the curve $y^2 = ax - x^2$; whence $2y\dot{y} = a\dot{x} - 2x\dot{x}$, and $\frac{y}{\dot{y}} = \frac{2ax-2x^2}{a\dot{x}-2x\dot{x}}$, and therefore P H $= \frac{\overline{2ax-2x^2} \times \sqrt{\dot{x}^2+\dot{v}^2}}{a\dot{x}-2x\dot{x}}$: which will be expressed independent of fluxions, when the property of the curve B P $h$, or the relation of $x$ and $v$ is given: thus, let B P $h$ be the common parabola, and A B its *latus rec-*

*tum;* then $v$ being $=\sqrt{ax}$, $\dot{v}$ will be $=\frac{a\dot{x}}{2\sqrt{ax}}$, $\dot{x}^2+\dot{v}^2=\dot{x}^2+\frac{a\dot{x}^2}{4x}=\frac{\dot{x}^2\times\overline{4x+a}}{4x}$; and therefore P H $\left(\frac{\overline{2ax-2x^2}\times\sqrt{\dot{x}^2+\dot{v}^2}}{a\dot{x}-2x\dot{x}}\right)=\frac{\overline{a-x}\times\sqrt{4x^2+ax}}{a-2x}$.

Thus far relates to curves whose ordinates are parallel to each other: we come now to curves of the spiral kind, whose ordinates all issue from a point: such as the spiral B A G, whose ordinates C B, C A, C G, are all referred to the point C, called the center of the spiral.

### Illustration.

60. Let S A N be a tangent to the spiral at any point A, also let C T be perpendicular thereto, and let the arch CBA (considered as variable by the motion of A towards G) be denoted by $z$, and the ordinate C A by $y$. Then $\dot{z}:\dot{y}::$ A C $(y)$ : A T $=\frac{y\dot{y}}{\dot{z}}$.* Hence, if upon C A, as a diameter, a semi-circle be described, and in it, from A, a right-line A T equal to $\frac{y\dot{y}}{\dot{z}}$ be inscribed, that right-line will be a tangent to the spiral at the point A.



* Art. 5 & 35.

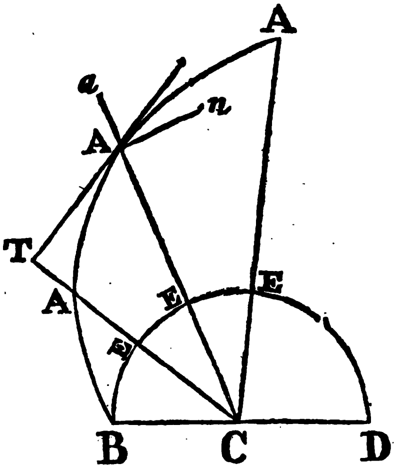
## EXAMPLE I.

61. Let the nature of the curve C B A be such that the arch C B A may be, always, to its cor-

responding ordinate C A in a constant ratio; namely as $a$ to $b$: then, because $z : y :: a : b$, we have $z = \frac{ay}{b}$, $\dot{z} = \frac{a\dot{y}}{b}$, and consequently A T $\left(\frac{y\dot{y}}{\dot{z}}\right) = \frac{by}{a} = \frac{b}{a} \times$ A C: therefore, AC and AT being in a constant ratio, the angle C A T must also be invariable. Which is a known property of the logarithmic spiral.

## EXAMPLE II.

62. Let B A A be the spiral of *Archimedes*; whose nature is such that the part E A of the generating ordinate, intercepted by the spiral and a circle B E D described about the same center C, is always in a constant ratio to the corresponding arch B E of that circle.



Suppose A $n$ perpendicular to A C, *&c.*

Put $BC = c$, $CA = y$, and let the given ratio of A E to B E, be that of $b$ to $c$: then $b : c :: y - c$ (AE) : $\frac{cy - c^2}{b} =$ B E: whose fluxion therefore is $= \frac{c\dot{y}}{b}$. Now if the right-line C E A $a$ be supposed to revolve about the center C, the angular celerity of the generating point A, in the perpendicular direction A $n$, will be to that of E as AC to EC; therefore as the latter of these celerities is expressed by $\frac{c\dot{y}}{b}$*, the former will be expressed by $\frac{y}{c} \times \frac{c\dot{y}}{b}$, or $\frac{y\dot{y}}{b}$: which is to $(\dot{y})$ the celerity of A, in the direction A $a$, as $\frac{y}{b}$ to unity, or as $y$ to $b$. Therefore C T and A T are in the same ratio (by *Art.* 35), and consequently $AC : CT :: \sqrt{y^2+b^2} : y$; and $AC : AT :: \sqrt{y^2+b^2} : b$; whence C T and A T are given equal to $\frac{y^2}{\sqrt{y^2+b^2}}$, and $\frac{by}{\sqrt{y^2+b^2}}$ respectively. From either of which the tangent A T may be drawn by *Art.* 60. And, in the same manner may the position of the tangent of any other spiral be determined.

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* Art. 5.

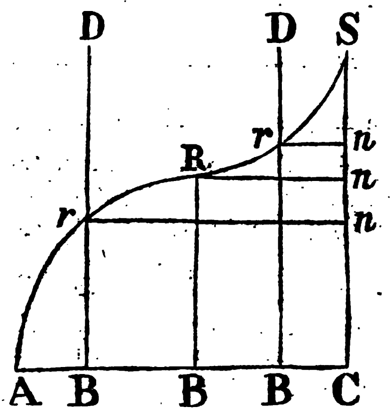
## SECTION IV.

### *Of the Use of Fluxions in determining the Points of Retrogression, or contrary Flexure in Curves.*

63. WHEN a curve A R S is, in one part A R concave, and in the other part R S convex, towards its axis A C, the point R limiting the two parts is called a point of retrogression, or contrary flexure. The manner of determining which will appear from the following

### ILLUSTRATION.

Suppose a right-line B D to be carried along uniformly, parallel to itself, from A towards C; and let the point $r$ so move in that line, at the same time, as to trace out, or describe, the given curve line A R S.



Then (*by Art.* 48) while the celerity of the point $r$, in the line B D, decreases, the curve will be concave to its axis A C; but when it increases, convex to the same: therefore, as any quantity is a *minimum* at the end of its decrease and the beginning of its increase,* it follows that the said celerity, at the point of inflexion R, must be a *minimum:* whence, if the fluxion of the ordinate B $r$, expressing that celerity,† be (as usual) denoted by $\dot{y}$; then will $\ddot{y}$ (the fluxion of $\dot{y}$) be equal to nothing in that circumstance.‡

So far relates to curves which are, in the former part concave, and in the latter convex, to their axes: but if (on the contrary) the celerity of $r$ first increases, and then decreases, that celerity at the required point, between the increase and decrease, will be a *maximum*; and therefore its fluxion (or $\ddot{y}$) is *likewise* equal to nothing in this case.§

Furthermore, if C S (perpendicular to A C) be now considered as an axis, and the abscissa S $n$ (or its complement B $r = y$) be supposed to flow uniformly, (as AB was supposed before); then, by the same argument, the second fluxion ($-\ddot{x}$) of the ordinate $n\,r$

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* Art. 22.

† Art. 5.

‡ Art. 22.

§ Art. 22.

(or its complement A B $=x$) will be equal to nothing. Hence it is evident that, at the point of contrary flexure, the second fluxion of the ordinate will become equal to nothing, if the abscissa be made to flow uniformly; and *vice versâ.*

## EXAMPLE I.

64. Let the nature of the curve A R S *(see the preceding figure)* be defined by the equation $ay=a^{\frac{3}{2}}x^{\frac{1}{2}}+x^2$ (the abscissa A B and the ordinate B $r$ being, as usual represented by $x$ and $y$ respectively). Then $\dot{y}$, expressing the celerity of the point $r$, in the line B D, will be equal to $\frac{\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x}+2x\dot{x}}{a}$: whose fluxion, or that of $\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}}+2x$ (because $a$ and $\dot{x}$ are constant) must be equal to nothing;* that is, $-\frac{1}{4}a^{\frac{3}{2}}x^{-\frac{3}{2}}\dot{x}+2\dot{x}=0$: whence $a^{\frac{3}{2}}x^{-\frac{3}{2}}=8$, $a^{\frac{3}{2}}=8x^{\frac{3}{2}}$, $64x^3=a^3$, and $x=\frac{1}{4}a=$ A B; therefore B R $\left(=\frac{a^{\frac{3}{2}}x^{\frac{1}{2}}+x^2}{a}\right)=\frac{9}{16}a$: from which the position of the point R is given.

## EXAMPLE II.

65. Let the nature of the proposed curve be defined by the equation $ay^2-a^2x-x^3=0$.

Then, by taking the first and second fluxions thereof (supposing $\dot{x}$ constant) we shall also have $2ay\dot{y}-a^2\dot{x}-3x^2\dot{x}=0$, and $2a\dot{y}^2+2ay\ddot{y}-6x\dot{x}^2=0$; whereof the latter, when $\ddot{y}$ is $=0$, becomes $2a\dot{y}^2-6x\dot{x}^2=0$, and therefore $\dot{y}^2=\frac{3x\dot{x}^2}{a}$: but by the former $\dot{y}=\frac{a^2\dot{x}+3x^2\dot{x}}{2ay}$; whence $\frac{3x\dot{x}^2}{a}=\frac{\overline{a^2\dot{x}+3x^2\dot{x}}|^2}{\overline{2ay}|^2}$, and consequently $12axy^2$

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* Art. 63.

$=\overline{a^2+3x^2}|^2$; but, by the given equation, $12axy^2=12a^2x^2+12x^4$, therefore $12a^2x^2+12x^4=\overline{a^2+3x^2}|^2$ or $3x^4+6a^2x^2-a^4=0$: whence $x$ will be found $=\frac{a}{\sqrt[4]{12}}(\sqrt{3}-1)$.

*Otherwise.*

Since $ay^2=a^2x+x^3$, we have $y=\frac{\overline{a^2x+x^3}|^{\frac{1}{2}}}{\sqrt{a}}$, and therefore $\dot{y}=\frac{\overline{\frac{1}{2}a^2\dot{x}+\frac{3}{2}x^2\dot{x}}\times\overline{a^2x+x^3}|^{-\frac{1}{2}}}{\sqrt{a}}$: whose fluxion, or that of $\overline{a^2+3x^2}\times\overline{a^2x+x^3}|^{-\frac{1}{2}}$ (because $\dot{x}$ is constant) being put $=0$, we get $6x\times\overline{a^2x+x^3}|^{-\frac{1}{2}}+\overline{a^2+3x^2}\times\overline{-\frac{1}{2}a^2-\frac{3}{2}x^2}\times\overline{a^2x+x^3}|^{-\frac{3}{2}}=0$, or $6x\times\overline{a^2x+x^3}+\overline{a^2+3x^2}\times-\frac{\overline{a^2+3x^2}}{2}$: whence $3x^4+6a^2x^2-a^4=0$, and $x=\frac{a}{\sqrt[4]{12}}(\sqrt{3}-1)$, the same as before.

## EXAMPLE III.

66. Let the proposed curve be the conchoid of *Nicomedes*, whereof the equation is $x^2y^2=\overline{a+y}|^2\times\overline{b^2-y^2}$,* or $x^2=\frac{\overline{a+y}|^2\times\overline{b^2-y^2}}{y^2}$.

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* Art. 57.

Here we have $x\dot{x} = \dfrac{\overline{\dot{y} \times \overline{a+y} \times \overline{b^2-y^2} - y\dot{y} \times \overline{a+y}|^2 \times y^2}}{y^4}$ $\dfrac{\overline{-y\dot{y} \times \overline{a+y}|^2 \times \overline{b^2-y^2}}}{y^4} = -\dfrac{\overline{a+y} \times \overline{ab^2+y^3}}{y^3}\,\dot{y} =$ $\overline{\dfrac{-a^2b^2}{y^3} - \dfrac{ab^2}{y^2} - a - y} \times \dot{y}$: whence, making $\dot{y}$ invariable, we also have $\dot{x}^2 + x\ddot{x} = \overline{\dfrac{3a^2b^2}{y^4} + \dfrac{2ab^2}{y^3} - 1} \times \dot{y}^2$: which, because $\ddot{x}$ is $=0$*, will be $\dot{x}^2 = \overline{\dfrac{3a^2b^2}{y^4} + \dfrac{2ab^2}{y^3} - 1} \times \dot{y}^2 = \dfrac{3a^2b^2 + 2ab^2y - y^4}{y^4} \times \dot{y}^2$. But since, by the former equation, $x\dot{x} = -\dfrac{\overline{a+y} \times \overline{ab^2+y^3}}{y^3} \times \dot{y}$, we likewise get $\dot{x}^2 = \dfrac{\overline{a+y}|^2 \times \overline{ab^2+y^3}|^2}{x^2y^6} \times \dot{y}^2$, and consequently $\overline{3a^2b^2+2ab^2y-y^4} \times x^2y^2 = \overline{a+y}|^2 \times \overline{ab^2+y^3}|^2$: but, by the equation of the curve $x^2y^2$ is $= \overline{a+y}|^2 \times \overline{b^2-y^2}$; therefore $\overline{3a^2b^2+2ab^2y-y^4} \times \overline{a+y}|^2 \times \overline{b^2-y^2} = \overline{a+y}|^2 \times \overline{ab^2+y^3}|^2$, and $\overline{3a^2b^2+2ab^2y-y^4} \times \overline{b^2-y^2} = \overline{ab^2+y^3}|^2$; whence $y^4 + 4ay^3 + 3a^2y^2 - 2ab^2y - 2a^2b^2 = 0$; which divided by $y+a$, gives $y^3 + 3ay^2 - 2ab^2 = 0$; from whence $y$ may be determined. But if $b=a$, the equation will become more simple by dividing again by $y+a$; in which case we get $y^2 + 2ay - 2a^2 = 0$, and consequently $y = a\sqrt{3} - a$.

## EXAMPLE IV.

67. Let $a^4y = 180a^3x^2 - 110a^2x^3 + 30ax^4 - 3x^5$.

Then will $a^4\dot{y} = 360a^3x\dot{x} - 330a^2x^2\dot{x} + 120ax^3\dot{x} - 15x^4\dot{x}$;

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* Art. 63.

And $a^4\dddot{y} = 360a^3\dot{x}^2 - 660a^2x\dot{x}^2 + 360ax^2\dot{x}^2 - 60x^3\dot{x}^2$.

Therefore, $6a^3 - 11a^2x + 6ax^2 - x^3 = 0$:*

Which being divisible by any one of the three quantities $a-x$, $2a-x$, or $3a-x$, the root $x$ must therefore have three values, $a$, $2a$, and $3a$, and consequently the curve, defined by the given equation, as many points of contrary flexure.

But, if you would know whether the part of the curve lying between any two adjacent points, thus found, be convex or concave towards the axis; see whether the value of the expression for the second fluxion of the ordinate, between the two corresponding roots, be positive or negative: for, in the former case, the curve is convex, and in the latter concave,† (provided the *whole* curve lies on the same side the axis). Thus, in the example before us; because the second fluxion of the ordinate is always as $6a^3 - 11a^2x + 6ax^2 - x^3$ $(= \overline{a-x} \times \overline{2a-x} \times \overline{3a-x})$ and it appears that the value of this expression, while $x$ is less than the first root $a$, will be positive; the curve, therefore, at the beginning, will be convex to its axis: but when $x$ becomes greater than $a$, the said expression being negative, the curve will then be concave, and so continue till $x$ is equal to the second root $2a$; after which the fluxion again becoming affirmative, the curve will accordingly be convex till $x = 3a$; beyond which limit the curvature continually tends the same way.

But it will be proper to observe, that there are cases where the second fluxion of the ordinate may become equal to nothing, without either changing its value from positive to negative, or the contrary (similar to those already taken notice of in *Sect.* II. *p.* 45 and 46), which cases always happen when the equation admits of an even number of equal roots: and then the point found as above is not a point of inflexion, because the curvature on either side of it tends the same way.

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* Art. 63.

† Art. 5 & 48.

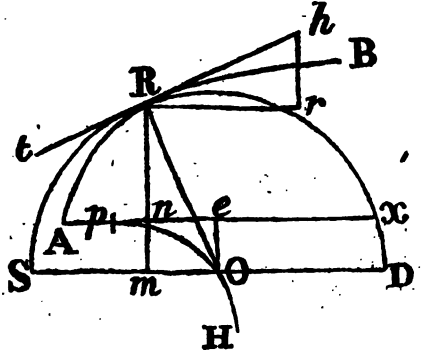
## SECTION V.

### *The Use of Fluxions in determining the Radii of Curvature, and the Evolutes of Curves.*

68. A CURVE $pOH$ is said to be the evolute of another curve $ARB$, when it is of such a nature, that a thread $ROH$, coinciding therewith (or wrapped upon the same) being unwound or disengaged from it, by a power acting at the end $R$, shall, by that end (the thread continuing tight) describe the given curve $ARB$.

### ILLUSTRATION.

From the point $O$, where the right-line $RO$ (called the radius of curvature) touches the evolute $pOH$,



let the semi-circle $SRD$ be described; which semi-circle, having the same radius with the given curve, at $R$, will consequently have the same degree of curvature.——But the curvature in two curves is the same, when, the fluxion of their abscissas being the same, both the first, and second fluxions of their corresponding ordinates R *n* and R *m* are respectively equal to each other: for, the first fluxions being equal, the two curves will have, at the common point R, one and the same tangent *t* R *h*:* and, if the second fluxions be likewise equal, the curvature, or deflection from that tangent, will also be the same in both; because these last express the increase or decrease of motion in the direction of the ordinate,† upon which the curvature entirely depends.‡

* Art. 48.
† Art. 19.
‡ Art. 48.

This being premised, let the abscissa S*m* of the semicircle (considered as variable) be put $=w$, its ordinate R$m=v$, R $r=\dot w$, $r\,h=\dot v$, and R $h=\dot z$: then, R *h* being a tangent to the circle at R,‖ the triangles R *h r* and R O *m* will be equiangular, and therefore $\dot w$ (R *r*) : $\dot z$ (R *h*) :: $v$ (R*m*) : R O $=\frac{v\dot z}{\dot w}$; which, because the radius of every circle is a constant quantity, must be invariable, and consequently its fluxion $\frac{\dot v\dot z+v\ddot z}{\dot w}=0$: whence $v$ is found $=\frac{\dot v\dot z}{-\ddot z}=\frac{\dot z^2}{-\ddot v}$ (because, $\dot w$ being constant, and $\dot w^2+\dot v^2=\dot z^2$, we have, in fluxions, $2\dot v\ddot v=2\dot z\ddot z$, and so $\frac{\dot v\dot z}{-\ddot z}=\frac{\dot z^2}{-\ddot v}$). Therefore since $v$ is $=\frac{\dot z^2}{-\ddot v}$, we also get SO $=$ RO$\left(\frac{v\dot z}{\dot w}\right)=\frac{\dot z^3}{-\dot w\ddot v}=\frac{\overline{\dot v^2+\dot w^2}|^{\frac{3}{2}}}{-\dot w\ddot v}$: which last is a general expression for the radius of any circle, whatever, in terms of the fluxions of its abscissa *(w)* and ordinate *(v)*. But, by what is premised above, these fluxions are respectively equal to those of the abscissa A *n (x)* and ordinate R *n (y)* of the proposed curve ARB. Therefore, by writing $\dot x$, $\dot y$, and $\ddot y$, instead of $\dot w$, $\dot v$, and $\ddot v$, we have $\frac{\overline{\dot y^2+\dot x^2}|^{\frac{3}{2}}}{-\dot x\ddot y}$ $\left(=\frac{\dot z^3}{-\dot x\ddot y}\right)$ for the general value of the radius of curvature, R O.

‖ Art. 48.

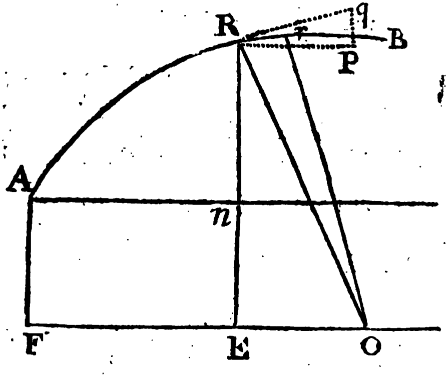
### *The same otherwise.*

If the radius of the circle be put$=$R, and every thing else be supposed as above; then (by the property of the circle) we shall have $v^2$ (R$m^2$) $= 2\,\mathrm{R}\,w - w^2$ (S$m \times$D$m$): whence in fluxions (making $\dot{w}$ constant) we get $2v\dot{v} = 2\mathrm{R}\dot{w} - 2w\dot{w}$, and $2\dot{v}^2 + 2v\ddot{v} = -2\dot{w}^2$: from the last of which equations $v$ is found$=\frac{\dot{v}^2+\dot{w}^2}{-\ddot{v}}$ $=\frac{\dot{z}^2}{-\ddot{v}}$; and consequently R O$\left(\frac{v\dot{z}}{\dot{w}}\right)=\frac{\dot{z}^3}{-\dot{w}\ddot{v}} = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *the same as before.*

### *Otherwise without the Circle.*

Let R O and $r$ O be two rays perpendicular to the curve, indefinitely near to each other; and from their intersection O, let OF be drawn parallel to A$n$, cutting R$n$ and A F (parallel to R$n$) in E and F.

Therefore, supposing R E$=v$, A$n=x$, R$n=y$, &c. (as before) we shall have, by similar triangles, as R P



$(\dot{x})$ : P$q$ $(\dot{y})$ : : RE $(v)$ : EO$=\frac{v\dot{y}}{\dot{x}}$; and consequently F O (A$n$+E O) $= x+\frac{v\dot{y}}{\dot{x}}$: which value (as well as that of AF) continuing the same whether we regard the radius R O, or the radius $r$ O, its fluxion must therefore be equal to nothing; that is, $\dot{x} + \frac{\overline{\dot{v}\dot{y} + v\ddot{y}} \times \dot{x} - v\dot{y}\ddot{x}}{\dot{x}^2} = 0$; whence $v = \frac{\dot{x}^3 + \dot{x}\dot{v}\dot{y}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, and consequently R O $\left(\frac{v\dot{z}}{\dot{x}}\right) = \frac{\dot{x}^2\dot{z} + \dot{v}\dot{y}\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{x}^2\dot{z} + \dot{y}^2\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$: which, if $\dot{x}$ is supposed constant, or $\ddot{x} = 0$, will become $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *as above.*

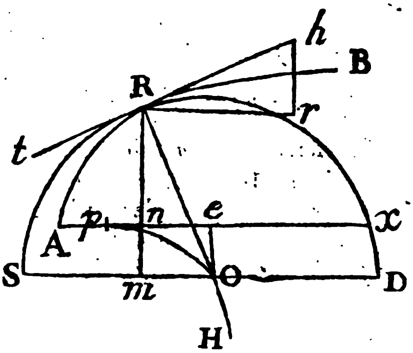
But if $\dot{y}$ be supposed constant, it will be $\frac{\dot{z}^3}{\dot{y}\ddot{x}}$. And, if $\dot{z}$ be constant, it will then be $\frac{\dot{z}\dot{y}}{\ddot{x}}$: for, since $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$, by taking the fluxion thereof, we have $2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 0$; whence $\ddot{y} = -\frac{\dot{x}\ddot{x}}{\dot{y}}$; and therefore R O $\left(= \frac{\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}\right) = \frac{\dot{z}^3}{\dot{y}\ddot{x} + \frac{\dot{x}^2\ddot{x}}{\dot{y}}} = \frac{\dot{y}\dot{z}^3}{\overline{\dot{y}^2 + \dot{x}^2} \times \ddot{x}} = \frac{\dot{y}\dot{z}}{\ddot{x}}$, *as before.*

Now from the several values of the radius of curvature R O, found above, the corresponding values of A $e$ and $e$ O will likewise be given.

Thus, if $\dot{x}$ be made constant; then, R O being $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, we shall have A$e$ (A$n$ + O$m$ = A$n$ + $\frac{\dot{y}}{\dot{z}}$ × R O) $= x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}$, and $e$ O (R$m$ − R$n$ = $\frac{\dot{x}}{\dot{z}}$ × RO − R$n$) $= \frac{\dot{z}^2}{-\ddot{y}} - y$.

But, if $\dot{y}$ be made constant, then, R O being $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, we shall have A e $= x + \frac{\dot{z}^2}{\ddot{x}}$, and $e$ O $= \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x}} - y$.

Lastly, if $\dot{z}$ be supposed constant; then R O being $=\frac{\dot{y}\dot{z}}{\ddot{x}}$, we shall have $Ae=x+\frac{\dot{y}^2}{\ddot{x}}$, and $eO=\frac{\dot{x}\dot{y}}{\ddot{x}}-y$.



Which several expressions will serve as so many general theorems for determining the quantity of curvature, and the evolutes of given curves: but, before we proceed to examples, it will be proper to observe, that the right-line A $p$, denoting the radius of curvature at the vertex A (to be found by making $x$, or $y$, $=0$) must always be subtracted from R O and A $e$, to have the true length of the arch $p$ O, and its corresponding abscissa $p\,e$.

## EXAMPLE I.

69. Let the given curve ARB be the common parabola, whose equation is $y=a^{\frac{1}{2}}x^{\frac{1}{2}}$: Then will $\dot{y}=\frac{1}{2}a^{\frac{1}{2}}\dot{x}x^{-\frac{1}{2}}$ $=\frac{a^{\frac{1}{2}}\dot{x}}{2x^{\frac{1}{2}}}$, and (making $\dot{x}$ constant) $\ddot{y}=-\frac{1}{2}\times\frac{1}{2}a^{\frac{1}{2}}\dot{x}^2x^{-\frac{3}{2}}$ $=\frac{-a^{\frac{1}{2}}\dot{x}^2}{4x^{\frac{3}{2}}}$: whence $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2})=\frac{\dot{x}}{2}\sqrt{\frac{4x+a}{x}}$,

and the radius of curvature R O $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right)=\frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}}$: which at the vertex A, where $x=0$, will be $=\frac{1}{2}a=$ A $p$. Moreover A $e$ $\left(x+\frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}\right)=\frac{1}{2}a+3x$, and therefore $p$ $e$ (A $e$ $-$ A $p$) $=3x$, the abscissa of the evolute: likewise O $e$ $\left(\frac{\dot{z}^2}{-\ddot{y}}-y\right)=\frac{4x^{\frac{3}{2}}}{\sqrt{a}}$ the ordinate of the evolute. Therefore, $\overline{\text{O}\,e}|^2\times a$ being in a constant ratio to $\overline{p\,e}|^3$, namely as 16 to 27, the curve is, in this case, the semi-cubical parabola: whose arch $p$ O (R O $-$ A $p$) is also given $=\frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}}-\frac{1}{2}a$.

## EXAMPLE II.

70. Let the curve A R B denote a parabola of any other kind: then, because $y=ax^n$ is an equation to all kinds of parabolas, we have $\dot{y}=nax^{n-1}\dot{x}$ and $\ddot{y}=n\times\overline{n-1}\times ax^{n-2}\dot{x}^2$: therefore $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2})=\dot{x}\sqrt{1+n^2a^2x^{2n-2}}$, RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right)=\frac{\overline{1+n^2a^2x^{2n-2}}|^{\frac{3}{2}}}{-n\times\overline{n-1}\times ax^{n-2}}$, A $e$ $\left(x+\frac{\dot{y}\dot{z}^2}{-\ddot{y}}\right)=x-\frac{x+n^2a^2x^{2n-1}}{n-1}$, O $e$ $\left(\frac{\dot{z}^2}{-\ddot{y}}-y\right)$ $=\frac{1+\overline{2n-1}\times na^2x^{2n-2}}{-\overline{n-1}\times nax^{n-2}}$, and A $p=-\frac{n^2a^20^{2n-1}}{n-1}$: which, if $n=\frac{1}{2}$, will become $=\frac{a^2}{2}$; but, if $n$ be greater than $\frac{1}{2}$, it will be $=0$; and, if $n$ be less than $\frac{1}{2}$,

it will be infinite: whence it appears, that the radius of curvature at the vertex will be a finite quantity in curves whose first (or least) ordinates are in the subduplicate ratio of their abscissas, and in all other cases, either nothing or infinite.

## EXAMPLE III.

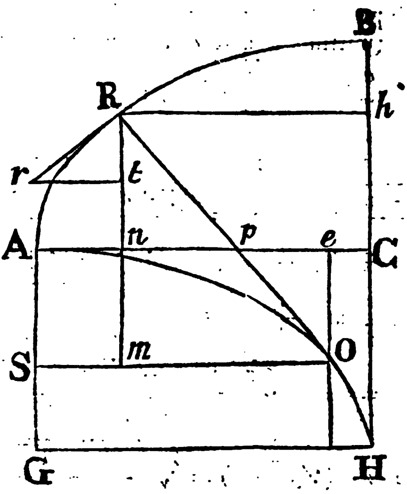
71. Suppose the given curve to be an ellipsis; whose equation (putting $a$ and $c$ for the two principal diameters) is $a^2y^2 = c^2 \times \overline{ax - x^2}$.

Here, by taking the first and second fluxions of the given equation, we have $2a^2y\dot{y} = c^2\dot{x} \times \overline{a - 2x}$, and $2a^2\dot{y}^2 + 2a^2y\ddot{y} = c^2\dot{x} \times -2\dot{x} = -2c^2\dot{x}^2$; whence $\dot{y} = \dfrac{c^2\dot{x} \times \overline{a-2x}}{2a^2y}$, and $-\ddot{y} = \dfrac{a^2\dot{y}^2 + c^2\dot{x}^2}{a^2y}$: which, by substituting the values of $y$ and $\dot{y}$, will become $\dot{y} = \dfrac{c\dot{x} \times \overline{a-2x}}{2a\sqrt{ax-x^2}}$, and $-\ddot{y} = \dfrac{a^2c^2\dot{x}^2 \times \overline{a-2x}|^2}{4a^2 \times \overline{ax - xx} \times ac\sqrt{ax - x^2}}$ $+ \dfrac{c\dot{x}^2}{a\sqrt{ax-x^2}} = \dfrac{c\dot{x}^2}{a} \times \dfrac{\overline{a-2x}|^2 + 4 \times \overline{ax-x^2}}{4 \times \overline{ax-x^2}\sqrt{ax-x^2}} = \dfrac{ca\dot{x}^2}{4 \times \overline{ax-x^2}|^{\frac{3}{2}}}$: therefore $\dot{z}$ $(\sqrt{\dot{y}^2 + \dot{x}^2}) = \sqrt{\dfrac{c^2\dot{x}^2 \times \overline{a - 2x}|^2}{4a^2 \times \overline{ax-x^2}} + \dot{x}^2}$ $= \dfrac{\dot{x}}{2a}\sqrt{\dfrac{c^2a^2 + \overline{a^2 - c^2} \times \overline{4ax - 4x^2}}{ax - x^2}}$, and the radius of curvature $\left(\dfrac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \dfrac{\overline{a^2c^2 + \overline{a^2 - c^2} \times \overline{4ax - 4x^2}}|^{\frac{3}{2}}}{2a^4c}$: which when the diameters $a$ and $c$ are equal, or the ellipsis degenerates to a circle, will be every where equal to $\dfrac{\overline{a^2c^2}|^{\frac{3}{2}}}{2a^4c}$, or $\frac{1}{2}a$; agreeable to the definition of a circle.

## EXAMPLE IV.

72. *To find the Radius of Curvature, and the Evolute of the Common Cycloid.*

Let ARB be the given curve, and AOH its evolute; also let R$h$ and OS be parallel to AC, and $e$O and R$m$



perpendicular to A C, and put A R B ($=2$ B C) $=a$, A R $=z$, A $n=x$, and R $n=y$: then B R $=a-z$, B$h$ $=\frac{1}{2}a-y$; and, by the property of the curve, $a^2$ (A B$^2$) : $\overline{a-z}|^2$ (B R$^2$) :: $\frac{1}{2}a$ (B C) : $\frac{1}{2}a-y$ (B $h$) whence $y=\frac{2az-z^2}{2a}$; therefore $\dot{y}=\frac{a\dot{z}-z\dot{z}}{a}$, $\dot{z}^2-\dot{y}^2$ $(\dot{x}^2)=\frac{\overline{2az-z^2}\times\dot{z}^2}{a^2}$, and $\dot{x}=\frac{\dot{z}\sqrt{2az-z^2}}{a}$. Whence (making $\dot{z}$ constant) $\ddot{x}=\frac{\dot{z}^2\times\overline{a-z}}{a\sqrt{2az-z^2}}$; from which we get R O, or A O $(=\frac{\dot{y}\dot{z}}{\ddot{x}})$[*] $=\sqrt{2az-z^2}$, and $e$ O, or A S $(=\frac{\dot{y}\dot{x}}{\ddot{x}}-y)=\frac{2az-z^2}{2a}$; which, when $z=a$, or ROH coincides with BH, become AOH (BH) $=a$, and CH (AG) $=\frac{1}{2}a$. Hence, because it appears that, A H² $(a^2)$ : A O² $(2az-z^2)$ :: A G $(\frac{1}{2}a)$ : A S $(\frac{2az-z^2}{2a})$ it follows that the evolute A O H is also a cycloid equal, and similar, to the involute A R B.

If the evolute had been given, or supposed, a cycloid, and the involute required, the process would have been, more simple, as follows:

Let A H (2A G) $=a$, AO (= R O) $=z$, A S $=x$, SO $=y$, BR $=v$, B$h=w$, R$r=\dot{v}$, R$t=\dot{w}$, &c. Then it will be,†

$$\dot{y}:\dot{z}\ (::\text{O}m:\text{O R})::\text{R}t\ (\dot{w}):\text{R}r=\frac{\dot{w}\dot{z}}{\dot{y}}.$$

$$\dot{z}:\dot{y}::z\ (\text{R O}):\text{O}m=\frac{z\dot{y}}{\dot{z}},$$

$$\dot{z}:\dot{x}::z\ (\text{R O}):\text{R}m=\frac{z\dot{x}}{\dot{z}},$$

Whence we have $\dot{v}=\frac{\dot{w}\dot{z}}{\dot{y}}$, R$n$ (R$m$ − A S) $=\frac{z\dot{x}}{\dot{z}}-x$, and A$n$ (O S − O$m$) $=y-\frac{z\dot{y}}{\dot{z}}$; which expressions answer to any curve whatever.

But, in the case above proposed, A H² $(a^2)$ : A O² $(z^2)$ :: AG $(\frac{1}{2}a)$ : AS $(x)$; therefore $x=\frac{z^2}{2a}$, $\dot{x}=\frac{z\dot{z}}{a}$, and $\dot{y}$ $(\sqrt{\dot{z}^2-\dot{x}^2})=\frac{\dot{z}\sqrt{a^2-z^2}}{a}$; and consequently R$n$ $(\frac{z\dot{x}}{\dot{z}}-x)=\frac{z^2}{a}-\frac{z^2}{2a}=\frac{z^2}{2a}=\frac{1}{2}a-w$ (or C B − B$h$):

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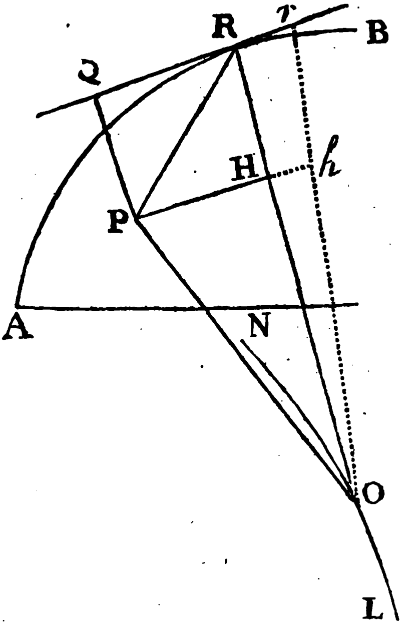
[*] Art. 68.

† Art. 48.

whence also $w = \frac{a^2 - z^2}{2a}$, and $\dot{v}\ \left(\frac{\dot{w}\,\dot{z}}{\dot{y}}\right) = \frac{a\dot{w}}{\sqrt{a^2 - z^2}} = \frac{a\dot{w}}{\sqrt{2aw}}$: therefore it will be $\dot{v} : \dot{w}\ (:: a : \sqrt{2aw})$ $:: \sqrt{\tfrac{1}{2}a} : \sqrt{w}$; that is, as $\mathrm{R}r : \mathrm{R}t :: \sqrt{\mathrm{BC}} : \sqrt{\mathrm{B}h}$: which is a known property of the cycloid.

Hitherto regard has been had to curves where the ordinates are parallel to each other; but when the ordinates are all referred to a given point, as in spirals, &c. other theorems will become necessary; and may be thus derived.

73. Let A R B be the proposed curve, P the point, or center, to which its ordinates are referred, N O L the evolute, and R O the ray of curvature at R: moreover, let PH be perpendicular to RO; and, supposing the ordinate PR $(y)$ to become variable by the motion of the point R along the curve, let the fluxions of AR and P H $(p)$, expressing the celerities of the points R and H in directions perpendicular to RO*, be denoted by $\dot{z}$ and $\dot{p}$ respectively.



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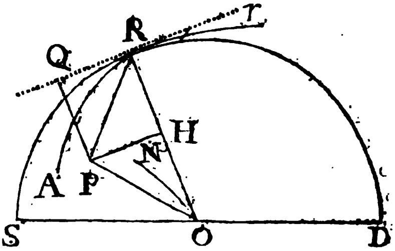
* Art. 5.

Therefore, the celerities of any two points, in a right-line revolving about a center, being as the distances from that center, it follows that $p : \dot{z} :: \text{OH} : \text{OR}$; whence by division (putting RH $= v$) we have $\dot{z}-\dot{p} : \dot{z} :: v\,(\text{RH}) : \text{RO} = \frac{v\dot{z}}{\dot{z}-\dot{p}} = \frac{vp\dot{z}}{p\dot{z}-p\dot{p}}$: but $p\dot{z} = y\dot{y}$ (*by Art.* 60) and therefore $\text{R O} = \frac{vy\dot{y}}{y\dot{y}-p\dot{p}}$; which, because $y^2-p^2$ is $= v^2$ (and therefore $y\dot{y}-p\dot{p} = v\dot{v}$) will also be $= \frac{vy\dot{y}}{v\dot{v}} = \frac{y\dot{y}}{\dot{v}}$.

*The same otherwise.*

Let S R D be a circle described about the point O, as a center, and suppose the distance P R to be variable by the motion of the point R along the arch of the circle (instead of the curve): then, drawing O P, and putting OR $= r$, PR $= y$, *&c.* as before, we shall get $\text{O P}^2$ $(\text{OR}^2+\text{PR}^2-2\,\text{OR}\times\text{RH}) = r^2+y^2-2rv$; which (as well as $r$) being a constant quantity, its fluxion $2y\dot{y}-2r\dot{v}$ must be equal to nothing; and therefore $r = \frac{y\dot{y}}{\dot{v}}$, the very same as above. Nor is it of any consequence whether $\dot{y}$ and $\dot{v}$ be here looked upon as respecting the circle, or the curve; since, at R, they must be the same in both cases; otherwise the curvature could not be the same*. Now from the value of R O thus found, which (corrected, when necessary) will also express the length of the arch N O of the evolute†, the ordinate P O and the tangent O H of the evolute



* Art. 68.

† Art. 68.

may be easily deduced. For O H ( R O − R H ) $= \frac{y\dot{y}}{\dot{v}} - v = \frac{p\dot{p}}{\dot{v}}$, and P O $(= \sqrt{OH^2 + PH^2}) = \frac{p\sqrt{\dot{p}^2 + \dot{v}^2}}{\dot{v}}$; whence the nature of the evolute is known.

## EXAMPLE I.

74. Let the given curve A R be the logarithmic spiral, whose nature is such, that the angle P R Q (or R P H) which the ordinate makes with the curve is every where the same.

Then (denoting the sine of that angle by $b$, and the radius of the tables by $a$) we have R H $(v) = \frac{by}{a}$, and therefore R O $\left(\frac{y\dot{y}}{\dot{v}}\right) = \frac{ay\dot{y}}{b\dot{y}} = \frac{ay}{b}$; which being to P R $(y)$ in the constant ratio of $a$ to $b$, or of P R to RH, the triangles R O P and R P H must therefore be similar, and so the angle P O H, which the ordinate P O makes with the evolute, being every where equal to P R Q, will likewise be invariable. Whence it appears that the evolute is also a logarithmic spiral, similar to the involute; and that a right-line drawn from the center, perpendicular to the ordinate, of any logarithmic spiral, will pass through the center of curvature.

## EXAMPLE II.

75. Let the curve proposed be the spiral of *Archimedes*; where we have $p = \frac{by}{\sqrt{y^2 + b^2}}$, and $v = \frac{y^2}{\sqrt{y^2 + b^2}}$ (see *Art.* 62). Therefore $\dot{v} = 2y\dot{y} \times \overline{y^2 + b^2}\big|^{-\frac{1}{2}} + y^2 \times$

$$-\tfrac{1}{2} \times 2y\dot{y} \times \overline{y^2+b^2}|^{-\frac{3}{2}} = \frac{2y\dot{y}}{\overline{y^2+b^2}|^{\frac{1}{2}}} - \frac{y^3\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}} =$$

$$\frac{2y\dot{y} \times \overline{y^2+b^2} - y^3\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}} = \frac{y^3\dot{y} + 2b^2y\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}}$$ ; whence the radius of *curvature $\frac{y\dot{y}}{\dot{v}}$ is here $= \frac{\overline{y^2+b^2}|^{\frac{3}{2}}}{y^2+2b^2}$; which being $= \frac{b}{2}$, when $y=0$, the arch of the evolute,† reckoned from the vertex, is therefore $= \frac{\overline{y^2+b^2}|^{\frac{3}{2}}}{y^2+2b^2} - \frac{b}{2}$.

After the very same manner you may proceed in other cases: but if the value of $\dot{v}$ (or $\frac{y\dot{y}}{\dot{v}}$) changes in any case, from positive to negative, the radius of curvature (R O) after becoming infinite, will fall on the other side of the tangent, and the corresponding point of the curve, when $\dot{v}=0$, will be a point of *Contrary-Flexure.* Whence it may be observed that the point of inflexion, in a curve whose ordinates are referred to a center, may be found by making the fluxion of the perpendicular, drawn from the center to the tangent, equal to nothing, which case is not taken notice of in the preceding Section.

* Art. 73.

† Art. 68.

## SECTION VI.

### *Of the Inverse Method, or the Manner of determining the Fluents of given Fluxions.*

76. IN the *Inverse Method*, which teaches the manner of finding the respective flowing quantities of given fluxions, there will be no great difficulty in conceiving the reasons, if what is already delivered in *Sect.* 1 *on the Direct Method*, has been duly considered: though the difficulties that occur in this part, upon another account, are indeed vastly superior.

It is an easy matter, or not impossible at most, to find the fluxion of any flowing quantity whatever; but in the *Inverse Method* the case is quite different: for, as there is no method for deducing the fluent from the fluxion, *a priori*, by a direct investigation, so it is impossible to lay down rules for any other forms of fluxions than those particular ones which we know, from the direct method, belong to such and such kinds of flowing quantities. Thus, for example, the fluent of $2x\dot{x}$ is known to be $x^2$, because it is found in Art. 6 and 14, that $2x\dot{x}$ is the fluxion of $x^2$; but the fluent of $y\dot{x}$ is unknown, since no expression has been discovered that produces $y\dot{x}$ for its fluxion.

77. Now, as the principal rule in the *Direct Method* is that for the fluxions of powers derived in Art. 8 (where it is proved that the fluxion of $x^n$ is *universally* expressed by $nx^{n-1}\dot{x}$); so the most general rule that can be given in the *Inverse Method*, must be that arising from the converse thereof; which *shows how to assign the fluent of any power of a variable quantity drawn into the fluxion of the root;* and which, expressed in words, will be as follows.

*Divide by the fluxion of the root, add unity to the exponent of the power, and divide by the exponent so increased.*

For, dividing the fluxion $nx^{n-1}\dot{x}$ by $\dot{x}$ (the fluxion of the root $x$) it becomes $nx^{n-1}$; and, adding 1 to the exponent $(n-1)$ we have $nx^n$; which, divided by $n$, gives $x^n$, the true fluent of $nx^{n-1}\dot{x}$, by Art. 8.

Hence (by the same rule) the

Fluent of $3x^2\dot{x}$ will be $= x^3$;

That of $8x^2\dot{x} = \frac{8x^3}{3}$;

That of $2x^5\dot{x} = \frac{x^6}{3}$;

That of $y^{\frac{1}{2}}\dot{y} = \frac{2}{3}y^{\frac{3}{2}}$;

That of $ay^{\frac{5}{3}}\dot{y} = \frac{3dy^{\frac{8}{3}}}{8}$;

That of $y^{\frac{m}{n}}\dot{y} = \frac{y^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \frac{ny^{\frac{m+n}{n}}}{m+n}$;

That of $\frac{a\dot{x}}{x^n}$, or $a\dot{x}x^{-n}$, $= \frac{ax^{1-n}}{1-n}$;

That of $\overline{a+z}|^3 \times \dot{z} = \frac{\overline{a+z}|^4}{4}$;

And that of $\overline{a^m+z^m}|^n \times z^{m-1}\dot{z} = \frac{\overline{a^m+z^m}|^{n+1}}{m\times\overline{n+1}}$:

for *here* the root, or the quantity under the general index $n$, being $a^m+z^m$, and its fluxion $= mz^{m-1}\dot{z}$ (Art. 14) we shall, by dividing by the last of these quantities, have $\frac{\overline{a^m+z^m}|^n}{m}$; whence, increasing the index by unity, and dividing by $(n+1)$ the index so increased, there comes out $\frac{\overline{a^m + z^m}|^{n+1}}{m \times \overline{n+1}}$.

After the very same manner the fluents of other expressions may be deduced, when the quantity, or multiplicator, without the vinculum, is either equal, or in a constant ratio, to the fluxion of the quantity under the vinculum: as in the expression $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$; where the number of dimensions of $z$ under the *vinculum* (or general index) being equal to those of $z$ without the *vinculum* + 1, the fluent may therefore be had, as in the preceding examples; and will come out $\frac{\overline{a+cz^n}|^{m+1} \times d}{nc \times \overline{m+1}}$: and, that this (or any other expression derived in like manner) is the true fluent will evidently appear, by supposing $x$ equal to $a + cz^n$ the quantity under the vinculum; for then (equal quantities having equal fluxions) $\dot{x}$ will be $= ncz^{n-1}\dot{z}$;* and consequently $\overline{a + cz^n}^m \times dz^{n-1}\dot{z}$ $\left(= x^m \times \frac{d\dot{x}}{nc}\right) = \frac{dx^m\dot{x}}{nc}$; whose fluent is therefore $\frac{dx^{m+1}}{nc \times \overline{m+1}}$ † $= \frac{d \times \overline{a+cz^n}|^{m+1}}{nc \times \overline{m+1}}$, *as before.*

* Art. 8.

† Art. 77.

78. In assigning the fluents of given fluxions, there is another particular that ought to be attended to, not yet taken notice of; and that is, whether the flowing quantity, found by the common rule above delivered, does not require the addition or subtraction of some constant quantity to render it complete. This indeed

can only be known from the nature of the problem under consideration; but that such an addition or subtraction may, in some cases, become necessary, is evident from the subject itself; since a flowing quantity increased or decreased by a constant quantity, has still the same fluxion; and therefore the fluent of that fluxion is as properly expressed by the whole compound expression, as by the variable part of it alone: thus, for instance, the fluent of $nx^{n-1}\dot{x}$ may be either represented by $x^n$ or by $x^n \pm a$, because ($a$ being constant) the fluxion of $x^n \pm a$, as well as of $x^n$, is $nx^{n-1}\dot{x}$.

79. Hence it appears that it is the variable part of a fluent only which is assignable by the common method; the constant part (when such becomes necessary) being to be ascertained from the particular nature of the problem. Now, to do this, the best way is to consider how much the variable part of the fluent, first found, differs from the truth in that particular circumstance, when the required quantity which the whole fluent ought to express, is equal to nothing; then that difference added to, or subtracted from, the said variable part, as occasion requires, will give the fluent truly corrected: for, since the difference of two quantities flowing with the same celerity (or having equal fluxions) is either nothing at all or *constantly* the *same*, the difference in that circumstance will likewise be the difference in all other circumstances: and therefore being added to the lesser quantity, or subtracted from the greater, both become equal.

80. To render what is above delivered as familiar as may be, I shall put down a few examples; in which the variable quantities represented by $x$ and $y$ are supposed to begin their existence together, or to be generated at the same time.

1. Let $\dot{y} = a^2x\dot{x}$; then the fluent, found as usual, will be $y = \frac{a^2x^2}{2}$; where taking $y = 0$, $\frac{a^2x^2}{2}$ also vanishes (because then $x=0$ by hypothesis): therefore the fluent requires no correction in this case.

2. Let $\dot{y} = \overline{a+x}\vert^{3} \times \dot{x}$: here we first have $y = \frac{\overline{a+x}\vert^{4}}{4}$; but when $y = 0$, then $\frac{\overline{a+x}\vert^{4}}{4}$ becomes $= \frac{a^4}{4}$ (since $x$, by hypothesis, is then $= 0$): therefore $\frac{\overline{a+x}\vert^{4}}{4}$ *always* exceeds $y$ by $\frac{a^4}{4}$; and so the fluent properly corrected will be $y = \frac{\overline{a+x}\vert^{4} - a^4}{4} = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$.

But the very same fluent may be otherwise found, without needing any correction: for the given equation ($\dot{y} = \overline{a+x}\vert^{3} \times \dot{x}$), by expanding $\overline{a+x}\vert^{3}$ is transformed to $\dot{y} = a^3\dot{x} + 3a^2x\dot{x} + 3ax^2\dot{x} + x^3\dot{x}$; whence $y = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$; the same *as above*.

Hence it appears that the fluent of an expression, found according to one form, may require a very different correction from the fluent of the same fluxion found according to another form.

3. Let $\dot{y} = \overline{a^2-x^2}\vert^{\frac{1}{2}} \times x\dot{x}$; then, first, $y = -\frac{\overline{a^2-x^2}\vert^{\frac{3}{2}}}{3}$; where taking $y = 0$, $-\frac{\overline{a^2-x^2}\vert^{\frac{3}{2}}}{3}$ becomes

$= -\frac{a^3}{3}$; therefore $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$ is too little by $\frac{a^3}{3}$; and so the fluent corrected will be $y = \frac{a^3}{3} - \frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$.

4. Let $\dot{y} = \overline{a^m+x^m}|^{n} \times x^{m-1}\dot{x}$: here we first have $y = \frac{\overline{a^m+x^m}|^{n+1}}{m \times \overline{n+1}}$; and making $y=0$, the latter part of the equation becomes $\frac{\overline{a^m}|^{n+1}}{m \times \overline{n+1}} = \frac{a^{mn+m}}{m \times \overline{n+1}}$; whence the equation, or fluent, truly corrected, is $y = \frac{\overline{a^m+x^m}|^{n+1} - a^{mn+m}}{m \times \overline{n+1}}$.

5. Lastly, let $\dot{y} = \overline{a + bx^m + cx^n}|^{p} \times \overline{mbx^{m-1}\dot{x} + ncx^{n-1}\dot{x}}$; then, in the first place, we have $y = \frac{\overline{a+bx^m+cx^n}|^{p+1}}{p+1}$; which, corrected as above, becomes

$$y = \frac{\overline{a+bx^m+cx^n}|^{p+1} - a^{p+1}}{p+1}.$$

**81.** Hitherto $x$ and $y$ are both supposed equal to nothing at the same time; but that will not always be the case in the solution of problems. Thus, for instance, though the sine and tangent of an arch are both equal to nothing when the arch itself is equal to nothing, yet the secant is then equal to the radius: it will be proper therefore to add an example or two wherein the value of $y$ is equal to nothing, when that of $x$ is equal to any given quantity $a$.

Let, then, the equation $\dot{y}=x^2\dot{x}$ be first proposed; whereof the fluent (first taken) is $y=\frac{x^3}{3}$; but when $y=0$, then $\frac{x^3}{3}=\frac{a^3}{3}$, by hypothesis; therefore the fluent, corrected, is $y=\frac{x^3-a^3}{3}$.

Again, let the proposed equation be $\dot{y}=-x^n\dot{x}$; then will $y=-\frac{x^{n+1}}{n+1}$; which, corrected, becomes $y=\frac{a^{n+1}-x^{n+1}}{n+1}$.

Lastly, let $y=\overline{c^3+bx^2}|^{\frac{1}{2}}\times x\dot{x}$; then, first $y=\frac{\overline{c^3+bx^2}|^{\frac{3}{2}}}{3b}$; and when $y=0$ and $x=a$, $\frac{\overline{c^3+bx^2}|^{\frac{3}{2}}}{3b}$ becomes $=\frac{\overline{c^3+ba^2}|^{\frac{3}{2}}}{3b}$. therefore the fluent corrected is $y=\frac{\overline{c^3+bx^2}|^{\frac{3}{2}}-\overline{c^3+ba^2}|^{\frac{3}{2}}}{3b}$.

82. All the examples hitherto given relate to such fluxions as involve one variable quantity only in each term, whose fluents are assignable from the converse of the first general rule, *in Section* 1. But, besides these, various other forms of fluxions may be proposed, involving two or more variable quantities, whose fluents may also be found by help of the other two general rules delivered in the same section.

Thus the fluent of $y\dot{x}+x\dot{y}$ is expressed by $xy$;* that of $\frac{y\dot{x}-x\dot{y}}{y^2}$ by $\frac{x}{y}$;† that of $a\dot{x}+x\dot{y}+y\dot{x}$ by $ax+xy$;‡ and that of $\overline{nx\dot{y}y^{n-1}+y^n\dot{x}-nax^{n-1}\dot{x}}\times\overline{y^nx-ax^n}\big|^{\frac{p}{m}}$ by

$$\frac{m\times\overline{y^nx-ax^n}\big|^{\frac{p+m}{m}}}{p+m}$$

: for, dividing (in the last case) by the fluxion of the root $y^nx-ax^n$,* which (by Art. 14 and 15) is $nxy^{n-1}\dot{y}+y^n\dot{x}-nax^{n-1}\dot{x}$, we first have $\overline{y^nx-ax^n}\big|^{\frac{p}{m}}$; whence, adding unity to the exponent $\frac{p}{m}$, and dividing by the exponent so increased, we get

$$\frac{\overline{y^nx-ax^n}\big|^{\frac{p}{m}+1}}{\frac{p}{m}+1}=\frac{m\times\overline{y^nx-ax^n}\big|^{\frac{p+m}{m}}}{p+m}$$

for the true fluent of the quantity proposed. But it seldom happens that these kinds of fluxions, which involve two different variable quantities in one term, and yet admit of known or perfect fluents, are to be met with in practice; I shall therefore take no further notice of them in this place, but refer the reader to the second part of the work, my design here being to insist only upon what is most general and useful in the subject; which brings me to further consider those forms of fluxions, involving one variable quantity only, that frequently occur in the solution of problems, whose fluents may (after proper transformation) be found by the rule already delivered in Art. 77.

* Art. 10.

† Art. 13.

‡ Art. 10.

* Art. 77

88. It has been already hinted, that if a fluxion of the binomial kind, as $\overline{a+cz^n}|^m \times dz^{n-1}\dot{z}$, has the index $(n-1)$ of the variable quantity $(z)$ without the *vinculum* $+1$, equal to $(n)$ the index of the same quantity under the *vinculum*, the fluent thereof may be then truly found by the forementioned rule. But the same observation may be farther extended to *those cases where the index without the vinculum increased by unity is equal to any multiple of that under the vinculum*; as in the expressions, $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$, $\overline{a+cz^n}|^m \times dz^{3n-1}\dot{z}$, $\overline{a+cz^n}|^m \times dz^{4n-1}\dot{z}$, &c. Whose fluents are thus determined.

Put $a+cz^n = x$, then will $z^n = \frac{x-a}{c}$, and $nz^{n-1}\dot{z}$

* Art. 8. $= \frac{\dot{x}}{c}$;* and therefore $z^{2n-1}\dot{z} = \frac{x-a}{c} \times \frac{\dot{x}}{nc} =$

$\frac{x\dot{x}-a\dot{x}}{nc^2}$; whence by substitution we get $\overline{a+cz^n}|^m \times$

$dz^{2n-1}\dot{z} = \frac{x^m \times d \times \overline{x\dot{x}-a\dot{x}}}{nc^2} = d \times \frac{x^{m+1}\dot{x} - ax^m\dot{x}}{nc^2}$:

whose fluent (by Art. 77) is therefore $= \frac{d}{nc^2} \times$

$\overline{\frac{x^{m+2}}{m+2} - \frac{ax^{m+1}}{m+1}}$; which, by restoring the value of

$x$, becomes $\frac{d}{nc^2} \times \overline{\frac{\overline{a+cz^n}|^{m+2}}{m+2} - \frac{a \times \overline{a+cz^n}|^{m+1}}{m+1}} =$

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2}\times\overline{\frac{a+cz^n}{m+2}-\frac{a}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2}\times$$

$$\overline{\frac{cz^n}{m+2}-\frac{a}{m+2\times m+1}}\text{; the true fluent of } \overline{a+cz^n}|^m\times$$
$dz^{2n-1}\dot{z}$.

Again, for the fluent of $\overline{a+cz^n}|^m \times dz^{3n-1}\dot{z}$, because $z^{n-1}\dot{z}=\frac{\dot{x}}{nc}$, and $z^n=\frac{x-a}{c}$, we have $z^{3n-1}\dot{z}$

$$(=z^{2n}\times z^{n-1}\dot{z})=\frac{\overline{x-a}|^2}{c^2}\times\frac{\dot{x}}{nc}=\frac{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}{nc^3};$$

whence, $\overline{a+cz^n}|^m$ being $=x^m$, we get $\overline{a+cz^n}|^m\times$

$$dz^{3n-1}\dot{z}=\frac{dx^m\times\overline{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}}{nc^3}=\frac{d}{nc^3}\times$$

$\overline{x^{m+2}\dot{x}-2ax^{m+1}\dot{x}+a^2x^m\dot{x}}$; whose fluent is therefore

$$=\frac{d}{nc^3}\times\overline{\frac{x^{m+3}}{m+3}-\frac{2ax^{m+2}}{m+2}+\frac{a^2x^{m+1}}{m+1}}=$$

$$\frac{dx^{m+1}}{nc^3}\times\overline{\frac{x^2}{m+3}-\frac{2ax}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^3}\times$$

$$\overline{\frac{\overline{a+cz^n}|^2}{m+3}-\frac{2a^2+2acz^n}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^3}\times$$

$$\left(\frac{c^2z^{2n}}{m+3}-\frac{2acz^n}{m+3\times m+2}+\frac{2a^2}{m+3\times m+2\times m+1}\right).$$

*Universally*, let $r$ denote any whole positive number whatever, and let the fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$ be required; then, by putting $a+cz^n=x$, and proceeding as above, our proposed fluxion is transformed to $\frac{dx^m\dot{x}}{nc^r}\times\overline{x-a}|^{r-1}$; which, expanding $\overline{x-a}|^{r-1}$ (by the Binomial Theorem) becomes $\frac{d}{nc^r}\times$

$$\overline{x^{m+r-1}\dot{x}-\overline{r-1}\times ax^{m+r-2}\dot{x}+\overline{r-1}\times\frac{r-2}{2}\times a^2x^{m+r-3}\dot{x}}$$

&c. whose fluent is therefore $=\frac{d}{nc^r}\times\overline{\frac{x^{m+r}}{m+r}-\frac{\overline{r-1}\times ax^{m+r-1}}{m+r-1}+\frac{\overline{r-1}\times\overline{r-2}\times a^2x^{m+r-2}}{2\times\overline{m+r-2}}}$, &c. $=$

$$\frac{dx^{m+1}}{nc^r}\times\overline{\frac{x^{r-1}}{m+r}-\frac{\overline{r-1}\times ax^{r-2}}{m+r-1}+\frac{\overline{r-1}\times\overline{r-2}\times a^2x^{r-3}}{2\times\overline{m+r-2}}}$$

&c.

Where, $r$ being a whole positive number, the multiplicators $1$, $r-1$, $\overline{r-1}\times\overline{r-2}$, $\overline{r-1}\times\overline{r-2}\times\overline{r-3}$, &c. will therefore become equal to nothing, after the $r$ first terms; and so, the series terminating, the fluent itself will be truly exhibited in that number of terms: except when $m+r$ is likewise a whole positive number, less than $r$; in which circumstance, the divisors $m+r$, $m+r-1$, $m+r-2$, &c. becoming equal to nothing, before the multiplicators, the corresponding terms of the series will be infinite. And in that case the fluent is said to fail, since nothing can then be determined from it.

84. Besides the foregoing, there is another way of deriving the fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, in terms of the original flowing quantity $z$; which will afford a theorem more commodious for practice than that above given: the method of investigation is thus:

Let $d \times \overline{a+cz^n}|^{m+1} \times (Az^p + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}$ &c.) (where $p$, $v$, A, B, C, &c. denote unknown, but determinate quantities) be assumed for the fluent sought: then by taking the fluxion of the quantity so assumed, we shall have

$$dcn \times \overline{m+1} \times z^{n-1}\dot{z} \times \overline{a+cz^n}|^m \times \overline{Az^p + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}\ \&c.} + d \times \overline{a+cz^n}|^{m+1} \times \overline{pAz^{p-1}\dot{z} + \overline{p-v} \times Bz^{p-v-1}\dot{z} + \overline{p-2v} \times Cz^{p-2v-1}\dot{z},\ \&c.}$$

*which being put equal to the given fluxion, $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, and the whole equation divided by $\overline{a+cz^n}|^m \times dz^{-1}\dot{z}$, there comes out

$$\left.\begin{array}{l} + cn \times \overline{m+1} \times z^n \times \overline{Az^p + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}}\ \&c. \\ + \overline{a+cz^n} \times \overline{pAz^p + \overline{p-v} \times Bz^{p-v} + \overline{p-2v} \times Cz^{p-2v}}\ \&c. \end{array}\right\} = z^{rn}$$

whence, by collecting the coefficients of the like powers of $z$, we have

$$\left.\begin{array}{l} \left.\begin{array}{r} n \times \overline{m+1} \\ +p \end{array}\right\} \times cAz^{p+n} + \left.\begin{array}{r} n \times \overline{m+1} \\ +p-v \end{array}\right\} \times cBz^{p+n-v} + \left.\begin{array}{r} n \times \overline{m+1} \\ +p-2v \end{array}\right\} \times cCz^{p+n-2v}\ \&c. \\ \quad -z^{rn} \qquad + \qquad paAz^p \qquad + \overline{p-v} \times aBz^{p-v}\ \&c. \end{array}\right\} = 0$$

where, comparing $p+n$ and $rn$, the two greatest exponents of $z$, we find $p = rn - n = \overline{r-1} \times n$; and by comparing the two next inferior exponents $p+n-v$, and $p$,

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* Art. 8. 10.

we likewise get $v = n\dot{z}$ which values being substituted above, our equation is reduced to

$$\left.\begin{array}{l}\overline{m+r}\times ncAz^{rn}+\overline{m+r-1}\times ncBz^{rn-n}+\overline{m+r-2}\times ncCz^{rn-2n}\ \&c.\\ -z^{rn}+\quad\overline{r-1}\times naAz^{rn-n}+\overline{r-2}\times naBz^{rn-2n}\ \&c.\end{array}\right\}=0$$

where, putting $m+r=s$, and comparing the coefficients of the homologous terms,* we have $A=\frac{1}{snc}$, $B=-\frac{\overline{r-1}\times aA}{\overline{s-1}\times c}=-\frac{\overline{r-1}\times a}{s\times\overline{s-1}\times nc^2}$, $C=-\frac{\overline{r-2}\times aB}{\overline{s-2}\times c}=\frac{\overline{r-1}\times\overline{r-2}\times a^2}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}$, $D=-\frac{\overline{r-3}\times aC}{\overline{s-3}\times c}=-\frac{\overline{r-1}\times\overline{r-2}\times\overline{r-3}\times a^3}{s\times\overline{s-1}\times\overline{s-2}\times\overline{s-3}\times nc^4}$, &c. &c.

which values, with those of $p$ and $v$, being substituted in the assumed fluent, it becomes $d\times\overline{a+cz^n}|^{m+1}\times\overline{\frac{z^{rn-n}}{snc}-\frac{\overline{r-1}\times az^{rn-2n}}{s\times\overline{s-1}\times nc^2}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}}$ &c. $=\frac{d\times\overline{a+cz^n}|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}}$, &c. the true fluent of $\overline{a+cz^n}|^{m+1}\times dz^{rn-1}\dot{z}$, which was to be determined: which fluent therefore, when $r$ is a whole positive number, will always terminate in as many terms as are expressed by that number, except in that particular case specified in the last article. Thus, if $r=2$, or the given fluxion be $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$; then, $s$ $(m+r)$ being $= m+2$, the fluent itself will become

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* *Vide p.* 181 *of my Treatise of Algebra.*

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{nc\times\overline{m+2}} \times \overline{\frac{z^n}{1} - \frac{a}{\overline{m+1}\times c}} = \frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2} \times \overline{\frac{cz^n}{m+2} - \frac{a}{\overline{m+2}\times\overline{m+1}}};$$

which is exactly the same with the first of those found in Art. 83 by a different method.

The like agreement will likewise be found, when $r$ is $= 3$: but when $r$ either denotes a broken or a negative number, the series for the fluent will then run on to infinity; because no one of the multiplicators $r-1$, $r-2$, $r-3$, $r-4$, &c. can in that case be equal to nothing.

85. The foregoing fluent, it may be observed, was found by assuming $d\times\overline{a+cz^n}|^{m+1}\times\overline{Az^p+Bz^{p-v}+Cz^{p-2v}}$ &c. and comparing the two greatest exponents of the equation thence resulting: but if, instead of $Az^p+Bz^{p-v}+Cz^{p-2v}$ &c. an ascending series, as $Az^p+Bz^{p+v}+Cz^{p+2v}$ &c. (where the exponents of $z$ continually increase) be taken, and the two least indices of $z$ in the equation (in like manner resulting) be compared together, the same fluent will be had according to a different form, which will be of good use in many cases when the foregoing fails, or runs out into an infinite series.

Thus, if $p+v$, $p+2v$, &c. be wrote in the room of $p-v$, $p-2v$, &c. respectively, in the first equation of the last article, it will appear that

$$\left.\begin{array}{l} +cn\times\overline{m+1}\times z^n\times\overline{Az^p+Bz^{p+v}+Cz^{p+2v}}\ \&c.\\ +\overline{a+cz^n}\times pAz^p+\overline{p+v}\times Bz^{p+v}+\overline{p+2v}\times Cz^{p+2v}\ \&c.\end{array}\right\}=z^{rn}$$

which equation may be reduced to

$$\left.\begin{array}{l} paAz^p+\overline{p+v}\times aBz^{p+v}+\overline{p+2v}\times aCz^{p+2v}\ \&c.\\ -z^{rn}+\left.\begin{array}{c}n\times\overline{m+1}\\ +p\end{array}\right\}\times cAz^{p+n}+\left.\begin{array}{c}n\times\overline{m+1}\\ +p+v\end{array}\right\}\times cBz^{p+n+v}\ \&c.\end{array}\right\}=0$$

where, by comparing the two least exponents, &c. $p$ will be found $= rn$, $v = n$, $A = \frac{1}{pa} = \frac{1}{rna}$; $B =$

$$-\frac{\overline{p+n\times\overline{m+1}}\times cA}{\overline{p+v}\times a}=-\frac{\overline{r+m+1}\times ncA}{\overline{r+1}\times na}=-\frac{\overline{r+m+1}\times c}{r\times\overline{r+1}\times na^2};\ C=-\frac{\overline{p+v+n\times\overline{m+1}}\times cB}{\overline{p+2v}\times a}=-$$

$$\frac{\overline{r+m+2}\times ncB}{\overline{r+2}\times na}=\frac{\overline{r+m+1}\times\overline{r+m+2}\times c^2}{r\times\overline{r+1}\times\overline{r+2}\times na^3}\ \&c.\ \&c.$$

Therefore, denoting $r+m$ by $s$ (as above) the fluent of $\overline{a+cz^n}\big|^{m}\times dz^{rn-1}\dot{z}$ will *(also)* be truly represented by

$$d\times\overline{a+cz^n}\big|^{m+1}\times\left(\frac{z^{rn}}{rna}-\frac{\overline{s+1}\times cz^{rn+n}}{r\times\overline{r+1}\times na^2}+\frac{\overline{s+1}\times\overline{s+2}\times c^2z^{rn+2n}}{r\times\overline{r+1}\times\overline{r+2}\times na^3}\ \&c.\right)$$ or its equal $$\frac{\overline{a+cz^n}\big|^{m+1}\times dz^{rn}}{rna}\times\overline{1-\frac{\overline{s+1}\times cz^n}{\overline{r+1}\times a}+\frac{\overline{s+1}\times\overline{s+2}\times c^2z^{2n}}{\overline{r+1}\times\overline{r+2}\times a^2}\ \&c.}$$

Which series will terminate when $s$ (or $r+m$) is a whole negative number; and therefore in all such cases the fluent is exactly determined; provided $r$ be not also a negative integer less than $s$; for in this particular circumstance the fluent fails, the divisor first becoming equal to nothing. *Vide* Art. 83.

The use of the two foregoing general expressions, for the fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, will appear from the following examples.

## EXAMPLE I.

86. *Let it be required to find the Fluent of* $\dfrac{bx\dot{x}}{\overline{a+x}|^{\frac{1}{2}}}$, *or*

$$\overline{a+x}|^{-\frac{1}{2}} \times bx\dot{x}.$$

By comparing the fluxion here proposed with $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, we have $a=a$, $c=1$, $z=x$, $n=1$, $m=-\frac{1}{2}$, $d=b$, $rn-1$ (or $r-1$) $=1$; whence $r=2$, and $s$ $(r+m)$ $=\frac{3}{2}$; whereof the former being a whole positive number, let these values be therefore substituted in

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{snc} \times \left(\frac{z^{rn-n}}{1} - \frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c} + \right.$$

$$\left.\frac{\overline{r-1}\times\overline{r-1}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}, \text{ \&c.}\right)$$ the first of the two general expressions for the fluent, and it will become $\dfrac{b\times\overline{a+x}|^{\frac{1}{2}}}{\frac{3}{2}}\times\left(x-\dfrac{a}{\frac{1}{2}}\right)=\dfrac{b\times\overline{a+x}|^{\frac{1}{2}}\times\overline{2x-4a}}{3}$, the quantity sought in this case.

## EXAMPLE II.

87. *Let the Fluxion proposed be* $\frac{b\dot{x}x^{3n-1}}{\sqrt{a+fx^n}}$, or

$$\overline{a+fx^n}|^{-\frac{1}{2}} \times bx^{3n-1}\dot{x}.$$

Here, by proceeding as above, we have $a=a$, $c=f$, $z=x$, $n=n$, $m=-\frac{1}{2}$, $d=b$, $r=3$, and $s\ (r+m)=\frac{5}{2}$: whence, by substituting these several values in the same general expression, we get $\frac{b\times\overline{a+fx^n}|^{\frac{1}{2}}}{\frac{5}{2}nf}\times$

$$\overline{x^{2n}-\frac{2ax^n}{\frac{3}{2}f}+\frac{2a^2}{\frac{3}{2}\times\frac{1}{2}f^2}} = \frac{b\times\overline{a+fx^n}|^{\frac{1}{2}}}{nf^3} \times \frac{\overline{6f^2x^{2n}-8afx^n+16a^2}}{15}.$$

## EXAMPLE III.

88. *Wherein the Quantity proposed is* $\frac{\dot{y}\sqrt{g^2+y^2}}{y^6}$, or

$$\overline{g^2+y^2}|^{\frac{1}{2}}+y^{-6}\dot{y}.$$

Here we have $a=g^2$, $c=1$, $z=y$, $n=2$, $m=\frac{1}{2}$, $d=1$, $rn-1$ (or $2r-1$) $=-6$; whence $r$ $\left(=\frac{-6+1}{2}\right)$ $=-\frac{5}{2}$, and $s\ (r+m)=-2$; whereof the latter being a whole negative number, let the several values here exhibited be therefore substituted in

$$\frac{\overline{a+cz^n}|^{m+1}\times dz^{rn}}{rna}\times\left(1-\frac{\overline{s+1}\times cz^n}{\overline{r+1}\times a}+\frac{\overline{s+1}\times\overline{s+2}\times c^2z^{2n}}{\overline{r+1}\times\overline{r+2}\times a^2}\right.$$

&c.) the latter of the two general expressions above derived, and it will become $\frac{\overline{g^2+y^2}|^{\frac{3}{2}}\times y^{-5}}{-5g^2}\times$ $\left(1-\frac{-1\times y^2}{-\frac{3}{2}\times g^2}\right)=\frac{\overline{g^2+y^2}|^{\frac{3}{7}}\times\overline{2y^2-3g^2}}{15g^4y^5}$; the true fluent required.

## EXAMPLE IV.

89. *Lastly, let the given Fluxion be* $\overline{a-fz^n}|^{\frac{1}{2}}\times z^{-\frac{7}{2}n-1}\dot{z}$.

Then, $a$ being $=a$, $c=-f$, $m=\frac{1}{2}$, $d=1$, $r=-\frac{7}{2}$, and the rest as in the general fluxion $\overline{a+cz^n}|^m\times dz^{rn-1}\dot{z}$; we shall, by substituting in the second form (because $s$ is here equal to $(-3)$ a whole negative number) have $\frac{\overline{a-fz^n}|^{\frac{3}{2}}\times z^{-\frac{7}{2}n}}{-\frac{7}{2}na}\times\left(1-\frac{-2\times-fz^n}{-\frac{5}{2}a}\right.$

$$\left.+\frac{-2\times-1\times f^2z^{2n}}{-\frac{5}{2}\times-\frac{3}{2}a^2}\right)=\frac{\overline{a-fz^n}|^{\frac{3}{2}}}{-\frac{7}{2}naz^{\frac{7}{2}n}}\times\overline{1+\frac{4fz^n}{5a}+\frac{8f^2z^{2n}}{15a^2}}$$

$$=-\frac{\overline{a-fz^n}|^{\frac{3}{2}}\times\overline{30a^2+24afz^n+16f^2z^{2n}}}{105na^3z^{\frac{7}{2}n}}.$$

90. Having insisted largely on the manner of finding such fluents as can be truly exhibited in algebraic terms; it remains now to say something with regard to

those other forms of expressions, involving one variable quantity only, which, *yet*, are so affected by compound divisors and radical quantities, that their fluents cannot be *accurately* determined by any method whatsoever; of which there are innumerable kinds: but there is one general method whereby the fluents of such expressions are approximated, to any assigned degree of exactness; namely, the method of *Infinite Series;* which it will, therefore, be necessary to explain, so far as relates to the manner of expounding the value of any compound fraction, or surd quantity, by help of such a series.

## EXAMPLE I.

91. *Let, then, the Fraction* $\frac{ax}{a-x}$ *be first given; to be converted into an Infinite Series.*

Divide the numerator $ax$ by the denominator $a-x$, as is taught in compound division of common algebra; then the operation will stand as follows.

$$
\begin{array}{l}
a-x)\,ax \qquad \left(x+\dfrac{x^2}{a}+\dfrac{x^3}{a^2}+\dfrac{x^4}{a^3}+\&c.\right. \\
\qquad ax-x^2 \\
\qquad \overline{\quad +x^2 \quad} \\
\qquad\quad +x^2-\dfrac{x^3}{a} \\
\qquad\quad \overline{\qquad\quad +\dfrac{x^3}{a}} \\
\qquad\qquad\quad +\dfrac{x^3}{a}-\dfrac{x^4}{a^2} \\
\qquad\qquad\quad \overline{\qquad\qquad +\dfrac{x^4}{a}} \\
\qquad\qquad\qquad\qquad \&c.
\end{array}
$$

where the quotient, or series $x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\frac{x^5}{a^4}+\frac{x^6}{a^5}$ &c. infinitely continued, is taken to expound the value of the proposed fraction $\frac{ax}{a-x}$.

92. But, though the series thus arising ought to be carried on to an infinity of terms, to have the true value of the quantity first proposed; or, though the quotient, continued to ever so great a number of terms, will be *still* something defective of the truth; yet, if the value of the quantity $(x)$ in the numerator be but small in comparison of the quantity $(a)$ in the denominator, the remainder, after a few terms in the quotient, will become so exceeding small. as to be neglected without any considerable error; and then the value of the *whole*, or of the quantity first proposed will be very nearly exhibited, by taking a small number of the leading terms only.

Thus, for instance, let the value of $a$ be expounded by 10, and that of $x$ by unity; then the remainder $\left(\frac{x^3}{a}\right)$ after the two first terms of the quotient, being $=\frac{1}{10}$, this value, divided by the given divisor $(a-x=)$ 9, will therefore give $\frac{1}{90}=0{,}0\dot{1}11111$, &c. for the defect, by taking the two first terms only: but, if the three first terms be taken, the defect will be *still* less considerable; amounting to no more than $\frac{1}{900}$, or $0{,}00\dot{1}11111$, &c.

This may likewise be made to appear, without any regard to the remainder, by collecting into one sum, the values of all the terms to be taken: for, if only the first two $(x+\frac{x^2}{a})$ be proposed, their sum will be

$=1,1$; which, deducted from the true value of the given fraction $\frac{ax}{a-x}$ $(=\frac{10}{9})$ $=1,1111111$, &c. the difference will come out 0,01, *the very same as before.*

Thus, also, by collecting the sum of the three, four, and five, &c. first terms of the series, you will have 1,11; 1,111; and 1,1111, &c. which being successively deducted from 1,111111111, &c. (as above) there will remain 0,001111, &c. 0,0001111, &c. 0,00001111, &c. for the errors or defects in those cases respectively.

93. From what has been said in the preceding article it appears, that infinite series, in algebra (according to a common observation) are similar to, or correspond with, decimal fractions in common arithmetic: for, as a decimal fraction may be carried on to any proposed number of places, however great, and yet never amount to a quantity, which but a very little exceeds the value of the three or four first places; so a series may be infinite with regard to the number of its terms, and yet a few of the leading terms only, may be sufficient to express the value of the *whole*, very nearly: provided always, that the series has a sufficient rate of convergency, or that its terms decrease in a pretty large proportion: for otherwise, *even*, a great number of terms may be used to little purpose: thus, in the foregoing series, $x+\frac{x^2}{a}+\frac{x^3}{a^2}$, &c, if $x$ be taken $=a$, no number of terms will be sufficient to exhibit the value of the corresponding fraction $\frac{ax}{a-x}$, it being infinite in that circumstance.

94. Having endeavoured to show that the true value of an infinite series may be nearly obtained by adding together a few of the first terms only, I shall now proceed to give other examples of the manner of

converting fractional, and surd, quantities into such kinds of series, in order to the approximation of the fluents of expressions affected by them.

## EXAMPLE II.

*Let the quantity proposed be the fraction* $\frac{c^2}{c^2+2cy+y^2}$; then, by proceeding as in the first example, you will have

$$\begin{array}{l} c^2+2cy+y^2)\ c^2\ \ldots\ldots\ (1-\frac{2y}{c}+\frac{3y^2}{c^2}-\frac{4y^3}{c^3}\ \&c. \\ \qquad\qquad \underline{c^2+2cy+y^2} \\ \qquad\qquad -2cy-y^2 \\ \qquad\qquad \underline{-2cy-4y^2-\frac{2y^3}{c}} \\ \qquad\qquad\qquad +3y^2+\frac{2y^3}{c}\ \&c. \end{array}$$

Where, from a few of the first terms of the quotient, the law of continuation is manifest, the numerators being in arithmetical progression; and the signs, + and −, alternately.

## EXAMPLE III.

95. *Let the Quantity given be* $\frac{1+x^2-2x^4}{1-x-x^2}$.

Then the quotient will be $1+x+3x^2+4x^3+5x^4+9x^5+14x^6$ &c. where the *law of continuation* is manifest; being such that the co-efficient of each succeeding term is equal to the sum of those of the two terms immediately preceding it.

## EXAMPLE IV.

96. *Let the Radical Quantity* $\sqrt{a^2+x^2}$ *be proposed.*

Here, according to the common method of extracting the square root, the process will stand as fallows:

$$2a + \frac{x^2}{a} - \frac{x^4}{4a^3}\Big)\ a^2+x^2\ \Big(a + \frac{x^2}{2a} - \frac{x^4}{8a^3}\ \&c.$$

$$\begin{array}{l}
a^2 \\
\hline
+x^2 \\
+x^2+\dfrac{x^4}{4a^2} \\
\hline
-\dfrac{x^4}{4a^2} \\
-\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6} \\
\hline
+\dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6}\ \&c.\ \&c.
\end{array}$$

97. The law of continuation in series, thus arising, from radical quantities, is not easily discovered: but, if you would carry on the series to any proposed number of terms, the work will be a good deal shortened, by dividing the remainder by the divisor, when half that number of terms is found (as in common division) and observing, at the same time, to neglect all such terms whose indices would exceed the greatest, or the greatest plus the common difference, in the said remainder, according as the whole number of terms proposed to be found is odd, or even.

Thus, if it were proposed to continue the foregoing series $a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$ to 6 terms, then the divisor (or double quotient) being $2a+\frac{x^2}{a}-\frac{x^4}{4a^3}$, and the remainder $\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$ (as appears from the last article) the rest of the operation will stand thus:

$$\begin{array}{l}
2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\Big)\,\frac{x^6}{8a^4}-\frac{x^8}{64a^6}+0\,\Big(\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9} \\
\qquad\qquad\qquad\;\; \frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8} \\
\qquad\qquad\qquad\;\; \overline{\qquad\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad -\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8} \\
\qquad\qquad\qquad\qquad -\frac{5x^8}{64a^6}-\frac{5x^{10}}{128a^8} \\
\qquad\qquad\qquad\qquad \overline{\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad\quad +\frac{7x^{10}}{128a^8}
\end{array}$$

Which three terms thus found being added to those found above, we have $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^3}{128a^7}+\frac{7x^{10}}{256a^9}$, for the 6 first terms of an infinite series exhibiting the value of $\sqrt{a^2+x^2}$.

98. Another way of resolving any radical quantity, is to assume a series (with unknown co-efficients) for the value thereof; and then the series so assumed being raised to the second, third, or fourth power, &c. according as the root to be extracted is a square, cubic, or biquadratic one, &c. an equation will be obtained (free from surds) from whence, by comparing the homologous terms, the assumed co-efficients, and consequently the series sought, will be determined; as in

## EXAMPLE V.

*Where it is proposed to extract the Square Root of $a^{2n}+x^{2n}$ in an Infinite Series.*

In which case, assuming $A + Bx^{2n} + Cx^{4n} + Dx^{6n} + Ex^{8n}$ &c. for the required series, and taking the square thereof, we have

$$\left.\begin{array}{llllll} A^2 + 2ABx^{2n} & + 2ACx^{4n} & + 2ADx^{6n} & + 2AEx^{8n} & \&c. \\ & + B^2x^{4n} & + 2BCx^{6n} & + 2BDx^{8n} & \&c. \\ & & & + C^2x^{8n} & \&c. \end{array}\right\} = a^{2n} + x^{2n};$$

and consequently

$$\left.\begin{array}{llllll} A^2 & + 2ABx^{2n} & + 2ACx^{4n} & + 2ADx^{6n} & + 2AEx^{8n} & \&c. \\ -a^{2n} & - x^{2n} & + B^2x^{4n} & + 2BCx^{6n} & + 2BDx^{8n} & \&c. \\ & & & & + C^2x^{8n} & \&c. \end{array}\right\} = 0$$

Therefore $A^2 - a^{2n} = 0$, $2AB - 1 = 0$, $2AC + B^2 = 0$, $2AD + 2BC = 0$, $2AE + 2BD + C^2 = 0$, * &c. From which we get $A = a^n$; $B \left(= \frac{1}{2A}\right) = \frac{1}{2a^n}$; $C \left(= -\frac{B^2}{2A}\right) = -\frac{1}{8a^{3n}}$; $D \left(= -\frac{BC}{A}\right) = \frac{1}{16a^{5n}}$; $E \left(= -\frac{2BD + C^2}{2A}\right) = -\frac{5}{128a^{7n}}$ &c. whence we have $A + Bx^{2n} + Cx^{4n} + Dx^{6n}$ &c. $\left(= \sqrt{a^{2n} + x^{2n}}\right) = a^n$

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* Vide p. 181 of my Treatise of Algebra.

$+\frac{x^{2n}}{2a^n} - \frac{x^{4n}}{8a^{3n}} + \frac{x^{6n}}{16a^{5n}} - \frac{5x^{8n}}{128a^{7n}}$ &c. Which series, if $n$ be expounded by unity, will become $a + \frac{x^2}{a^2} - \frac{x^4}{8a^3}$ &c. the very same with that in the preceding article found by the common method.

## EXAMPLE VI.

99. *Let it be required to resolve* $\overline{a+bx^n}|^{\frac{1}{3}}$ *into an Infinite Series.*

Here, by assuming $A + Bx^n + Cx^{2n} + Dx^{3n}$ &c. and cubing the same, &c. we have

$$\left.\begin{array}{llll} A^3 + 3A^2Bx^n & + 3A^2Cx^{2n} & + 3A^2Dx^{3n} + & \&c. \\ -a - bx^n & + 3AB^2x^{2n} & + 6ABCx^{3n} + & \&c. \\ & & + \quad B^3x^{3n} + & \&c. \end{array}\right\} = 0$$

Therefore $A = a^{\frac{1}{3}}$; $B\left(=\frac{b}{3A^2}\right) = \frac{b}{3a^{\frac{2}{3}}}$; $C\left(= -\frac{B^2}{A}\right) = -\frac{b^2}{9a^{\frac{5}{3}}}$; $D\left(= -\frac{6ABC + B^3}{3A^2}\right) = \frac{5b^3}{81a^{\frac{8}{3}}}$ &c. and consequently, $\overline{a+bx^n}|^{\frac{1}{3}}$ $(= A + Bx^n + Cx^{2n} + \&c.)$

$$= a^{\frac{1}{3}} + \frac{bx^n}{3a^{\frac{2}{3}}} - \frac{b^2x^{2n}}{9a^{\frac{5}{3}}} + \frac{5b^3x^{3n}}{81a^{\frac{8}{3}}} + \&c.$$

And, in the same manner, may the root of any other quantity be extracted: but as the celebrated binomial theorem, discovered by the illustrious Sir *Isaac Newton*, is vastly more easy and expeditious, in raising powers and extracting roots than that, or any other, method, I shall now explain the uses thereof; but, first of all, it may not be amiss to show how the theorem itself, from the principles of fluxions, may be derived.

Let, then, $1+y$ be a binomial whose first term is unity, and its second term any proposed quantity $y$; and let the quantity to be expanded or thrown into a series be $\overline{1+y}|^{v}$; where the exponent $v$ is supposed to denote any number whatever, whole or broken, positive or negative.

Now it is evident that the first term of the required series must be unity; because when $y$ is $=0$, the other terms all vanish; and, in that case, $\overline{1+y}|^{v}$ is equal to unity. Let, therefore, $1 + Ay^{m} + By^{n} + Cy^{p} + Dy^{q}$ &c. be assumed to express the true value of the said series, or, which is the same, let

$\overline{1+y}|^{v} = 1 + Ay^{m} + By^{n} + Cy^{p} + Dy^{q}$ &c. where A, B, C, D, &c. $m$, $n$, $p$, $q$, &c. denote unknown, but determinate quantities:

Then, by taking the fluxion of the whole equation, (supposing $y$ variable) we shall have $v\dot{y} \times \overline{1+y}|^{v-1} = m\dot{y}\,Ay^{m-1} + n\dot{y}\,By^{n-1} + p\dot{y}\,Cy^{p-1} + q\dot{y}\,Dy^{q-1}$ &c. Whence, multiplying the sides of the two equations, cross-wise, and dividing by $\dot{y} \times \overline{1+y}|^{v-1}$, there comes out $\overline{1+y} \times \overline{mAy^{m-1} + nBy^{n-1} + pCy^{p-1} + qDy^{q-1}\ \&c.}$ $= v + vAy^{m} + vBy^{n} + vCy^{p} + vDy^{q}$ &c. which, by reduction, is

$$\left.\begin{array}{lllll} mAy^{m-1} & +nBy^{n-1} & +pCy^{p-1} & +qDy^{q-1} & \&c. \\ \quad * & +mAy^{m} & +nBy^{n} & +pCy^{p} & \&c. \\ -v & -vAy^{m} & -vBy^{n} & -vCy^{p} & \&c. \end{array}\right\} = 0$$

Now, since we are at liberty to take the exponents of $y$ what we will, so as to answer the conditions of the equation, or so that all the terms here put down may mutually destroy each other; let them, therefore, be so taken that the terms themselves may be homologous, that is, let $m-1=0$, $n-1=m$, $p-1=n$, $q-1=p$, &c. Then, $m$ being $=1$, $n=2$, $p=3$, $q=4$, &c. if these several values be substituted above, the equation itself will become

$$\left.\begin{array}{l} A+2By+3Cy^2+4Dy^3+\ \&c. \\ *+Ay\ \ +2By^2+3Cy^3\ \ \&c. \\ -v-vAy-vBy^2-vCy^3\ \ \&c. \end{array}\right\}=0$$

Where, taking $A-v=0$, $2B+A-vA=0$, $3C+2B-vB=0$, $4D+3C-vC=0$, &c. so that every column of homologous terms (and, consequently, the whole expression) may vanish, we also get $A=v$; $B\ (=\frac{vA-A}{2}=\frac{A\times\overline{v-1}}{2})=\frac{v\times\overline{v-1}}{2}$; $C\ (=\frac{vB-2B}{3}=B\times\frac{\overline{v-2}}{3})=v\times\frac{v-1}{2}\times\frac{v-2}{3}$; $D\ (=\frac{vC-3C}{4}=C\times\frac{v-3}{4})=v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}$, &c. &c.

Whence, by writing these values, with those of $m$, $n$, $p$, $q$, &c. in the series $1+Ay^m+By^n+Cy^p$ &c. first assumed, we, at length, find $\overline{1+y}|^v=1+vy+\frac{v}{1}\times\frac{v-1}{2}\times y^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times y^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times y^4+$ &c. which was to be investigated.

From the series here brought out, any power or root, of any other compound quantity, whether binomial, trinomial, &c. is easily deduced: for, if $p$ be put to represent the first term of any such quantity, and $Q$ the quotient of the rest of the terms divided by the first; then the quantity itself will be expressed by $P+PQ$, or $P\times\overline{1+Q}$, and the $v$ power thereof by $P^{v}\times\overline{1+Q}|^{v}$, which therefore is equal to $P^{v}\times(1+vQ+\frac{v}{1}\times\frac{v-1}{2}\times Q^{2}+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times Q^{3}+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times Q^{4}+$ &c.), by what is just now determined.

But when $v$ is a fraction, as in the notation of roots, the theorem here given will be rendered somewhat more commodious for practice, if, instead of $v$, a fraction as $\frac{m}{n}$ be substituted; by which means it will become $P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}=P^{\frac{m}{n}}\times(1+\frac{m}{n}Q+\frac{m}{n}\times\frac{m-n}{2n}Q^{2}+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}Q^{3}+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\times\frac{m-3n}{4n}Q^{4}+$ &c.) whose use, in converting radical quantities into infinite series will appear from the following Examples.

## EXAMPLE VII.

100. *Wherein it is proposed to extract the Square Root of $a^{2}+x^{2}$, in an Infinite Series.*

Here the quantity to be expanded being $\overline{a^{2}+x^{2}}|^{\frac{1}{2}}$, or $\overline{a^{2}}|^{\frac{1}{2}}\times\overline{1+\frac{x^{2}}{a^{2}}}|^{\frac{1}{2}}$, by comparing it with the general form, $P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}$, we have $P=a^{2}$, $Q=\frac{x^{2}}{a^{2}}$, $m=1$, and $n=2$: whence, by substituting these values in the last general equation, we get

$$\overline{a^2+x^2}|^{\frac{1}{2}} = a \times (1 + \tfrac{1}{2} \times \frac{x^2}{a^2} + \tfrac{1}{2} \times -\tfrac{1}{4} \times \frac{x^4}{a^4} + \tfrac{1}{2} \times -\tfrac{1}{4} \times -\tfrac{3}{6} \times \frac{x^6}{a^6} + \tfrac{1}{2} \times -\tfrac{1}{4} \times -\tfrac{3}{6} \times -\tfrac{5}{8} \times \frac{x^8}{a^8} + \text{\&c.}) = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}\ \text{\&c.}$$

Which series agrees exactly with those found in Art. 97 and 98, by different methods.

## EXAMPLE VIII.

101. *Let it be required to extract the Cube-Root of $b^3-y^3$, in an Infinite Series.*

Here by comparing $\overline{b^3}|^{\frac{1}{3}} \times \overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}$ $(=\overline{b^3-y^3}|^{\frac{1}{3}})$ with $P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}}$, it will be $P=b^3$, $Q=-\frac{y^3}{b^3}$, $m=1$, and $n=3$: therefore, by substitution, we get

$$\overline{b^3-y^3}|^{\frac{1}{3}} \left(=b \times \overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}\right) = b \times (1 + \tfrac{1}{3} \times -\frac{y^3}{b^3} + \tfrac{1}{3} \times -\tfrac{2}{6} \times \frac{y^6}{b^6} + \tfrac{1}{3} \times \tfrac{2}{6} \times -\tfrac{5}{9} \times -\frac{y^9}{b^9} + \tfrac{1}{3} \times -\tfrac{2}{6} \times -\tfrac{5}{9} \times -\tfrac{8}{12} \times \frac{y^{12}}{b^{12}} + \text{\&c.}) = b - \frac{y^3}{3b^2} - \frac{y^6}{9b^5} - \frac{5y^9}{81b^8} - \frac{10y^{12}}{243b^{11}}\ \text{\&c.}$$

## EXAMPLE IX.

102. *Let the Quantity to be converted into an Infinite Series be* $\frac{a}{\sqrt{ax-x^2}}$.

In this case the given quantity being first transformed to $\sqrt{\frac{a}{x}} \times \overline{1-\frac{x}{a}}\Big|^{-\frac{1}{2}}$ and $\overline{1-\frac{x}{a}}\Big|^{-\frac{1}{2}}$ afterwards compared with $\overline{1+Q}\Big|^{\frac{m}{n}}$, we have $Q = -\frac{x}{a}$, $m = -1$, and $n=2$; and therefore $\overline{1-\frac{x}{a}}\Big|^{-\frac{1}{2}}$ $(=\overline{1+Q}\Big|^{\frac{m}{n}} = 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-2n}{2n}Q^2 + \&c.)$ $1 + -\frac{1}{2} \times \frac{-x}{a} + -\frac{1}{2} \times -\frac{3}{4} \times \frac{x^2}{a^2} + -\frac{1}{2} \times -\frac{3}{4} \times -\frac{5}{6} \times \frac{-x^3}{a^3}$ &c. $= 1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4} + \&c.$ Which therefore, multiplied by $\sqrt{\frac{a}{x}}$, gives $\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{2a^{\frac{1}{2}}} + \frac{3x^{\frac{3}{2}}}{8a^{\frac{3}{2}}} + \frac{5x^{\frac{5}{2}}}{16a^{\frac{5}{2}}} + \frac{35x^{\frac{7}{2}}}{128a^{\frac{7}{2}}} + \&c. = \frac{a}{\sqrt{ax-x^2}}$, the quantity proposed.

103. It may not be improper to observe here, that, when both the terms of the proposed quantity are affirmative, and its exponent also affirmative and less than unity, the two first terms of the equal series will be positive, and the rest negative and positive, alternately; but if only the first term of the binomial be affirmative, all the terms of the series, after the first, will be negative: moreover, if the exponent of

the given quantity be negative, and both the terms affirmative, the signs will change alternately; but if only the first be affirmative, all the terms of the equal series will be positive.

## EXAMPLE X.

104. *Let the Quantity proposed be the Trinomial*

$$\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}}.$$

Here, by dividing the rest of the terms by the first, &c. our given quantity is reduced to $\overline{x^3}|^{\frac{1}{3}} \times \overline{1+2x+3x^2}|^{\frac{1}{3}}$. Therefore, in this case $P = x^3$, $Q = 2x + 3x^2$, $m = 1$, and $n = 3$: whence (by substitution) $\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}} = x \times (1 + {}^{1} \times \overline{2x+3x^2}| + \frac{1}{3} \times -\frac{2}{6} \times \overline{2x+3x^2}|^2 + \frac{1}{3} \times -\frac{2}{6} \times -\frac{5}{9} \times \overline{2x+3x^2}|^3$ &c.) $=$

$$x \times \overline{1 + \frac{2x + 3x^2}{3} - \frac{\overline{2x+3x^2}|^2}{9} + \frac{5 \times \overline{2x+3x}|}{81}} \text{ \&c.}$$

Which, reduced to simple terms, is $= x + \frac{2x^2}{3} + \frac{5x^3}{9} - \frac{68x^4}{81}$ &c.

105. When the proposed expression consists of a rational, multiplied by an irrational quantity, the series answering to the irrational one must be first found, and afterwards multiplied by the rational quantity: but, if two, or more, compound irrational quantities are to be drawn into each other, then take the series answering to each quantity, separately, and multiply them together; observing, always, to neglect all such terms whose indices would exceed that of the last, or highest,

term, which the series sought is proposed to be continued to.

## EXAMPLE XI.

106. *Let the Quantity proposed be* $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}}$.

First we have $\overline{1-x}|^{\frac{1}{10}} = 1 - \frac{x}{10} - \frac{9x^2}{10 \times 20} - \frac{9 \times 19x^3}{10 \times 20 \times 30} - \frac{9 \times 19 \times 29x^4}{10 \times 20 \times 30 \times 40} -$ &c. Which, multiplied by $1 + x$, produces $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}} = 1 + \frac{9x}{10} - \frac{29x^2}{10.20} - \frac{9.49x^3}{10.20.30} - \frac{9.19.69x^4}{10.20.30.40}$ &c. $= 1 + \frac{9x}{10} - \frac{29x^2}{200} - \frac{147x^3}{2000} - \frac{3933x^4}{80000} -$ &c.

## EXAMPLE XII.

107. *Where the Quantity to be expressed in an Infinite Series is* $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, or $\overline{a^2-x^2}|^{\frac{1}{2}} \times \overline{c^2-x^2}|^{-\frac{1}{2}}$.

Here we have, $\overline{a^2-x^2}|^{\frac{1}{2}}$ $(a \times \overline{1-\frac{x^2}{a^2}}|^{\frac{1}{2}} = a\,(\times$ $1 + \frac{1}{2} \times - \frac{x^2}{a^2} + \frac{1}{2} \times - \frac{1}{4} \times \frac{x^4}{a^4} + \frac{1}{2} \times - \frac{1}{4} \times - \frac{3}{6} \times - \frac{x^6}{a^6}$ $+$ &c.$) = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

And $\overline{c^2-x^2}|^{-\frac{1}{2}}$ $( = c^{-1} \times \overline{1 - \frac{xx}{cc}}|^{-\frac{1}{2}} = c^{-1} (\times 1 + -\frac{1}{2} \times -\frac{x^2}{c^2} + -\frac{1}{2} \times -\frac{3}{4} \times \frac{x^4}{c^4} + \&c.) = \frac{1}{c} + \frac{x^2}{2c^3} + \frac{3x^4}{8c^5} + \frac{5x^6}{16c^7}$ &c. Whence, multiplying these two values, one by the other, we get

$$\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 + \&c.$$

for the four first terms of the series sought.

## EXAMPLE XIII.

108. *Let the Quantity to be expounded be the Multinomial, or Infinite Series,* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v$; *whose Exponent* v *denotes any Number whatever, whole or broken, positive or negative.*

Here, dividing by the first term, the given quantity is transformed to $x^{pv} \times \overline{1 + ax^n + bx^{2n} + cx^{3n} + dx^{4n} + \&c.}|^v$; which, if $ax^n + bx^{2n} + cx^{3n}$ &c. be put $= y$, will become $x^{pv} \times \overline{1 + y}|^v$; which last expression (by Art. 99) is $= x^{pv} \times (1 + vy + \frac{v}{1} \times \frac{v-1}{2} \times y^2 + \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3} \times y^3 + \&c.)$ Whence (for brevity sake) putting $A = v$, $B = \frac{v}{1} \times \frac{v-1}{2}$, $C = \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = \frac{v}{1} \times$

$\frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, &c. and substituting for $y$, there comes out $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v =$ $x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^2} \times x^{pv+2n} +$ $\overline{Ac + 2Bab + Ca^3} \times x^{pv+3n} + \overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4}$ $\times x^{pv+4n} + (Ae + 2Bad + 2Bbc + 3Ca^2c + 3Cab^2 + 4Da^3b$ $+ Ea^5 \times x^{pv+5n} + \&c.)$

## EXAMPLE XIV.

109. *To extract the Square Root of $a^2 - x^2$, and from thence to determine the Fluent of $\dot{x}\sqrt{a^2 - x^2}$, in an Infinite Series.*

By proceeding as in the foregoing examples, the value of $\sqrt{a^2 - x^2}$ in an infinite series will be found to be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \&c.$ Which multiplied by $\dot{x}$ gives $\dot{x}\sqrt{a^2 - x^2} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$ &c. Whose fluent therefore (by Art. 77) is $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7} - \&c.$ Which was to be determined.

## EXAMPLE XV.

110. *Let it be required to approximate the Fluent of* $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}\times x^n\dot{x}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$ *in an Infinite Series.*

It appears, from example 12, that the value of $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, expressed in a series, is $\frac{a}{c}+\left(\frac{a}{2c^3}-\frac{1}{2ac}\right)\times x^2+\left(\frac{3a}{8c^5}-\frac{1}{4ac^3}-\frac{1}{8a^3c}\right)\times x^4+\left(\frac{5a}{16c^7}-\frac{3}{16ac^5}-\frac{1}{16a^3c^3}-\frac{1}{16a^5c}\right)\times x^6+$ &c. Which value being therefore multiplied by $x^n\dot{x}$, and the fluent taken (by the common method) we get $\frac{ax^{n+1}}{\overline{n+1}\times c}+\left(\frac{a}{2c^3}-\frac{1}{2ac}\right)\times\frac{x^{n+3}}{n+3}+\left(\frac{3a}{8c^5}-\frac{1}{4ac^3}-\frac{1}{8a^3c}\right)\times\frac{x^{n+5}}{n+5}+\left(\frac{5a}{16c^7}-\frac{3}{16ac^5}-\frac{1}{16a^3c^3}-\frac{1}{16a^5c}\right)\times\frac{x^{n+7}}{n+7}+$ &c.

## EXAMPLE XVI.

111. *Wherein it is proposed to approximate the Fluent of* $\overline{x^{p} + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^{v} \times x^{m-1}\dot{x}$ *in a Series.*

Here, if A be put $= v$, $B = v \times \frac{v-1}{2}$, $C = v \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = v \times \frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, the quantity $\overline{x^{p} + ax^{p+n} + bx^{p+2n} + cx^{p+3n}\ \&c.}|^{v}$ expanded, will be $= x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^{2}} \times x^{pv+2n} + \overline{Ac + 2Bab + Ca^{3}} \times x^{pv+3n} + (Ad + 2Bac + Bb^{2} + 3Ca^{2}b + Da^{4}) \times x^{pv+} + \&c.$ as appears from Art. 108. Therefore this expression being multiplied by $x^{m-1}\dot{x}$, and the fluent taken (as usual) we shall have $\frac{x^{pv+m}}{pv+m} + \frac{Aax^{pv+m+n}}{pv+m+n} + \frac{\overline{Ab+Ba^{2}} \times x^{pv+m+2n}}{pv+m+2n} + \frac{\overline{Ac + 2Bab + Ca^{3}} \times x^{pv+m+3n}}{pv + m + 3n} + \frac{\overline{Ad + 2Bac + Bb^{2} + 3Ca^{2}b + Da^{4}} \times x^{pv+m+4n}}{pv+m+4n} + \&c.$ for the quantity proposed to be found.

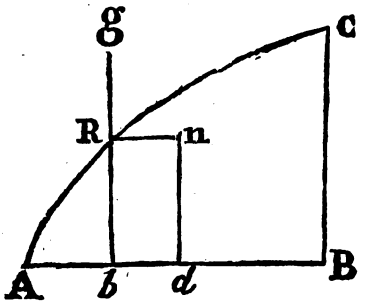
# SECTION VII.

## *Of the Use of Fluxions in finding the Areas of Curves.*

### CASE I.

112. *LET* A R C *be a Curve of any Kind whose Ordinates are perpendicular to an Axis* A B.

Imagine a right-line *b*R*g* (perpendicular to A B) to move parallel to itself from A towards B; and let the celerity thereof, or the fluxion of the abscissa A *b*, in any proposed position of that line, be denoted by *b d*; then it will appear from Art. 4. that the rectangle (*bn*) under *bd* and the ordinate *b* R, will express the corresponding fluxion of the generated area *a b* R; which fluxion, if A $b = x$, and $b$ R $= y$, will therefore be $= y\dot{x}$: from whence, by substituting for $y$ or $\dot{x}$ (according to the equation of the curve) and taking the fluent, the area itself will become known.

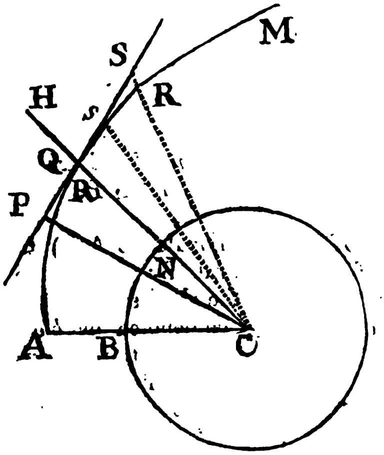


### CASE II.

113. *Let* A R M *be any Curve whose Ordinates* C R, C R *are all referred to a Point or Center.*

Conceive a right-line C R H to revolve about the given center C, and let a point R move along the said line, so as to trace out, or describe the proposed curve line A R M.

Now it is evident, that, if the point R was to move from any position Q, without changing its direction and velocity, it would proceed along the tangent Q S (instead of the curve) and describe areas Q $s$ C, Q S G about the center C, proportional to the times of their description; because those areas, or triangles, having the same altitude (CP), are as the bases Q $s$ and Q S, and these are as the times, because the motion in the tangent (upon that supposition) would be uniform.



Hence, if R S be taken to denote the value of ($\dot z$) the fluxion of the curve line A R, the corresponding fluxion of the area A R C, will be truly represented by the uniformly generated triangle Q C S*: which, putting the perpendicular (C P) drawn from the center to the tangent, $= s$, will therefore be $\left(= \frac{QS \times CP}{2}\right) = \frac{s\dot z}{2}$; from whence the area itself may be determined.

* Art. 2 & 5.

But, since in many cases the value of $\dot z$ cannot be computed (from the property of the curve) without some trouble, the two following expressions, for the fluxion of the area, will commonly be found more commodious, viz. $-\frac{sy\dot y}{2t}$ and $\frac{y^2\dot x}{2a}$, where $t = $ R P and $x =$ the arch B N of a circle, described about the center C, at any distance $a$ $(= CB)$. These expressions are derived from that above, in the following manner; viz. $\dot{z} : \dot{y} :: y$ (C R) : $t$ (R P) *; therefore $\dot{z} = \frac{y\dot{y}}{t}$; and consequently $\frac{s\dot{z}}{2} = \frac{sy\dot{y}}{2t}$; which is the first expression. * Art. 35.

Again, because the celerity of R in the direction of the tangent is denoted by $\dot{z}$, that in a direction perpendicular to C Q (whereby the point R revolves about the center C) will therefore be $(= \frac{CP}{CR} \times \dot{z})$ † $= \frac{s\dot{z}}{y}$; which being to $(\dot{x})$ the celerity of the point N (about the same center) as the distance (or radius) C R $(y)$ to the radius C N $(a)$ we shall, by multiplying extremes and means, have $\frac{as\dot{z}}{y} = y\dot{x}$; and consequently $\frac{s\dot{z}}{2} = \frac{y^2\dot{x}}{2a}$; which is the other expression. † Art. 35.

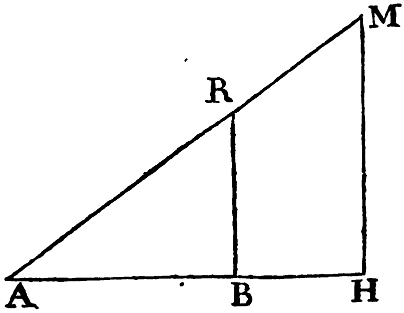
The method of applying this, together with the preceding forms, will appear at large from the following examples: wherein $x$, $y$, $z$, and $u$, are all along put to denote the abscissa, ordinate, curve-line, and the area respectively, unless where the contrary is expressly specified.

## EXAMPLE I.

114. *Let it be proposed to determine the Area of a right-angled Triangle* A H M.

Put the base A H $= a$, the perpendicular H M $= b$; and let A B $(x)$ be any portion of the base, considered as a flowing quantity, and let B R $(y)$ be the ordinate, or perpendicular, corresponding:

then, because of the similar triangles A H M and A B R, it will be $a : b :: x : y = \frac{bx}{a}$: whence $y\dot{x}$



* Art. 112. (the fluxion of the area A B R *) is, in this case, $= \frac{bx\dot{x}}{a}$; and consequently the fluent thereof, or the area

† Art. 77. itself $= \frac{bx^2}{2a}$:† which therefore, when $x = a$, and B R coincides with H M, will become $\frac{ab}{2} = \frac{\text{AH} \times \text{HM}}{2} =$ the area of the whole triangle A H M; which we also know from other principles.

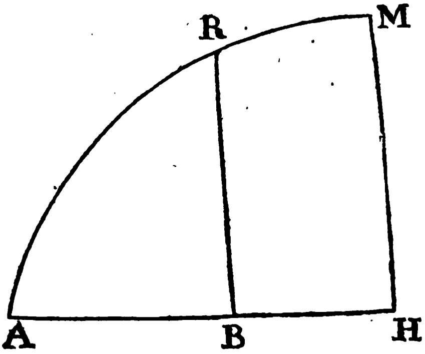
## EXAMPLE II.

115. *Let the Curve* ARMH, *whose Area you would find, be the common Parabola.*

In which case the relation of A B $(x)$ and B R $(y)$ being expressed by $y^2 = ax$ (where $a$ is the parameter)

‡ Art. 112. we thence get $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$; and therefore $\dot{u}$ $(= y\dot{x}$ ‡) $= a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$: whence $u = \frac{2}{3} \times a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}a^{\frac{1}{2}}x^{\frac{1}{2}} \times x = \frac{2}{3}yx$

(because $a^{\frac{1}{2}}x^{\frac{1}{2}}=y$) $=\frac{2}{3}\times AB\times BR$: hence *a parabola is* $\frac{2}{3}$ *of a rectangle of the same base and altitude.*



The area is here found in terms of $x$; but it will, many times, be more easily brought out in terms of $y$ (without radical quantities) as in the very case last proposed: where $x$ being $=\frac{y^2}{a}$, we therefore have $\dot{x}=\frac{2y\dot{y}}{a}$; and consequently $\dot{u}$ $(y\dot{x})=\frac{2y^2\dot{y}}{a}$: whence $u=\frac{2y^3}{3a}=\frac{2y}{3}\times\frac{y^2}{a}=\frac{2y}{3}\times x=\frac{2}{3}\times AB\times BR$; *the same as before.*

## EXAMPLE III.

116. *Let* ARM *(see the preceding figure) be a Parabola of any kind; whereof the general Equation is* $y^{m+n}=a^m x^n$.

Therefore, by extracting the root, or dividing each exponent by $m+n$, we have $y=a^{\frac{m}{m+n}}\times x^{\frac{n}{m+n}}$; whence

$\dot{u}\ (y\dot{x}) = a^{\frac{m}{m+n}} \times \dot{x}x^{\frac{n}{m+n}}$; and consequently $u$ (the true fluent, or area) $= a^{\frac{m}{m+n}} \times \dfrac{x^{\frac{n}{m+n}+1}}{\frac{n}{m+n}+1} =$

$$\frac{a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}} \times x \times \overline{m+n}}{m+2n} = \frac{m+n}{m+2n} \times yx = \frac{m+n}{m+2n} \times$$

AB × BR.

No notice has been yet taken of any constant quantity to be added to, or subtracted from, the variable one, first found, in order to render it complete, agreeable to the observation in Art. 78.

But that no such correction is required in any of the preceding examples, is evident from the nature of the figure; because, when $x$ and $y$ are nothing, the area ($u$) ought also to be nothing, which it actually is according to the equations above exhibited.

The fluent found in the succeeding example, will, however, stand in need of a correction.

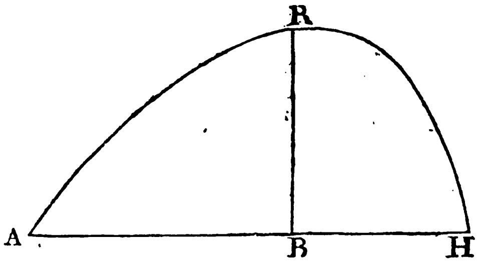
◆

## EXAMPLE IV.

117. *Where it is proposed to find the Area of the Curve* A R H, *whose Equation is* $x^4 - a^2x^2 + a^2y^2 = 0$.

Here, the given equation is reduced to $y = \dfrac{x \times \overline{a^2 - x^2}|^{\frac{1}{2}}}{a}$; whence $\dot{u}\ (= y\dot{x}) = \dfrac{\overline{a^2 - x^2}|^{\frac{1}{2}}}{a} \times x\dot{x}$: whereof the fluent (by the common rule*) is —

* Art. 77.



$\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$ : which, when $x=0$ and $u=0$, becomes $-\frac{a^2}{3}$; this therefore subtracted from $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$, leaves $\frac{a^2}{3}-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$ for the fluent corrected, or the true value of the area A B R.*

When the ordinate B R $\left(\frac{x\sqrt{a^2-x^2}}{a}\right)$ becomes equal to nothing, and B coincides with H, then $x$ will become $=a=$ AH ; and therefore the area of the whole curve A R H will be barely $=\frac{a^2}{3}=\frac{1}{3}\text{ A H}^2$.

## EXAMPLE V.

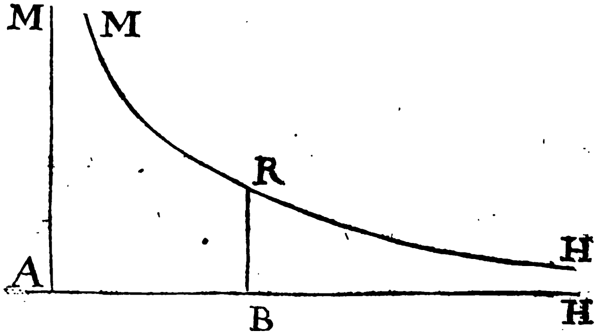
118. *Let it be required to determine the Area of the hyperbolical Curve whose Equation is* $x^m y^n = a^{m+n}$.

In this case we have $y=\frac{a^{\frac{m+n}{n}}}{x^{\frac{m}{n}}}=a^{\frac{m+n}{n}}\times x^{-\frac{m}{n}}$;

---

* Art. 78.

and therefore $\dot{u}$ ( $= y\dot{x}$) $= a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$ : whose fluent is $\frac{a^{\frac{m+n}{n}} \times x^{1-\frac{m}{n}}}{1-\frac{m}{n}} = \frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}$; which, when $x$ is



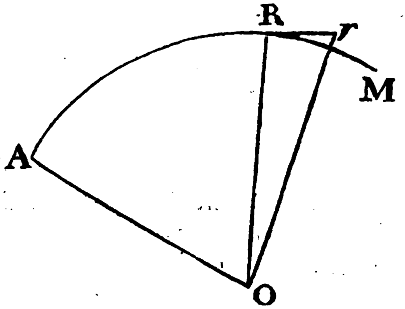
$=0$, will also be $=0$, if $n$ be greater than $m$: therefore the fluent requires no correction in this case; the area A M R B, included between the asymptote A M and the ordinate B R, being truly defined by $\left(\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}\right)$ the quantity above determined. But if $n$ be less than $m$, then the fluent, when $x=0$, will be infinite (because the index $\frac{n-m}{n}$ being negative, 0 becomes a divisor to $na^{m+n}$). Whence the area A M R B will also be infinite.

But here the area BRH comprehended between the ordinate, the curve, and the part BH of the other asymptote, is finite; and will be truly expounded by $\frac{na^{m+n} \times x^{\frac{n-m}{n}}}{m-n}$, the same quantity with its signs changed. For the fluxion of the part AMRB being $a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}}\dot{x}$ that of its supplement BRH must consequently be $-a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}}\dot{x}$: whereof the fluent is $-\frac{a^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{1-\frac{m}{n}}$, $=\frac{a^{\frac{m+n}{n}} \times x^{\frac{m-n}{n}}}{m-n}$ = the area BRH: which wants no correction; because, when $x$ is infinite, and the area BRH$=0$, the said fluent will also entirely vanish, seeing the value of $x^{\frac{m-n}{n}}$ (which is a divisor to $a^{\frac{m+n}{n}}$) is then infinite.

## EXAMPLE VI.

119. *Where let it be required to determine the Area of the circular Sector* AOR.

Then, putting the radius AO (or OR) $= a$, the



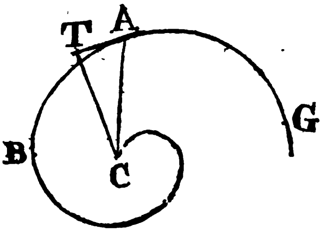
arch AR (considered as variable by the motion of R) $=z$, and R$r = \dot{z}$, the fluxion of the area will here

*Art. 113. be expressed by $\frac{a\dot{z}}{2}$ (= the triangle O R $r$ :*) whence the area itself is = $\frac{az}{2}$ = A O × ½ A R: from which it appears that the area of any circle is expressed by a rectangle under half the circumference and half the diameter.

## EXAMPLE VII.

120. *Wherein it is proposed to determine the Area* CBAC *of the logarithmic Spiral.*

Let the right-line A T touch the curve at A; upon which, from the center C, let fall the perpendicular C T: then, since by the nature of the curve the



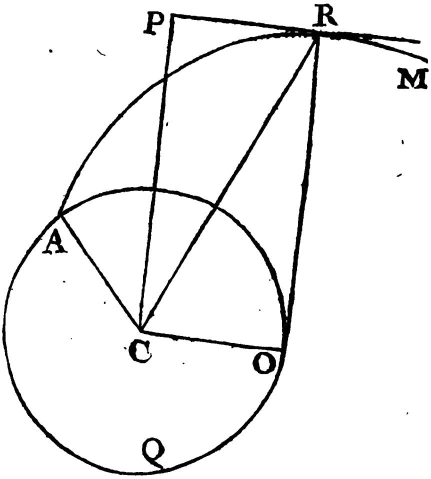
angle T A C is every where the same, the ratio of A T ($t$) to C T ($s$) will here be constant: and therefore the

†Art. 113. fluent of $\frac{s}{t} \times \frac{y\dot{y}}{2}$† = $\frac{s}{t} \times \frac{y^2}{4}$ = the area which was to be found.

## EXAMPLE VIII.

121. *Let the Curve* A R M *be the Involute of a given Circle* A O Q.

In which case the intercepted part of the tangent R P *(t)* being every where equal to the radius C O *(a)*



of the generating circle, we therefore have C P *(s)* $\sqrt{CR^2 - RP^2}) = \sqrt{y^2 - a^2}$: whence $\dot{u}$ ($= \frac{sy\dot{y}}{2t}$\*) $= \frac{\sqrt{y^2 - a^2} \times y\dot{y}}{2a}$; and consequently $u = \frac{\overline{y^2 - a^2}|^{\frac{3}{2}}}{6a} = \frac{CP^3}{6CA}$ = the required area A C R:

Which will also express the area A R O generated by the radius of evolution R O; because R O being =

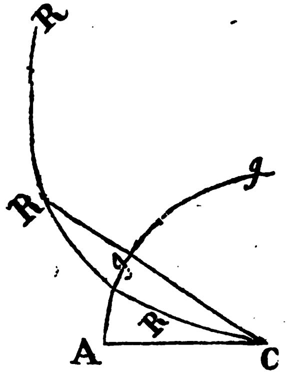
* Art. 113.

*Art. 119. the arch A O, the sector A′C O ($\frac{1}{2}$ A O × O C*) is equal to the triangle C R O ($\frac{1}{2}$ R O × O C) which equal quantities being successively subtracted from C A R O, there remains A O R = A C R.

## EXAMPLE IX.

122. *Let the Curve* C R R, *whose Area* C R *g* C, *you would find, be the Spiral of* Archimedes.

Let A C be a tangent to the curve at the center



C, about which center, with any radius A C ($=a$) suppose a circle A *g g* to be described; then the arch (or abscissa) A *g* corresponding to any proposed ordinate C R, being to that ordinate in a given or constant ratio (suppose as $m$ to $n$) we have $x$ (A g) $= \frac{my}{n}$;

* Art. 113. therefore $\dot{u} = \frac{y^2\dot{x}^*}{2a} = \frac{my^2\dot{y}}{2an}$, and consequently $u = \frac{my^3}{6an}$ = the area C R R *g* C

## EXAMPLE X.

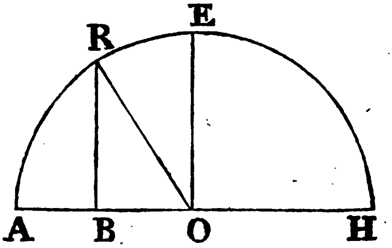
123. *Let the Equation of the Spiral* C R R *(see the last Figure) be* $x = by + cy^2 + dy^3 + ey^4 + fy^5 + \&c.$

Then, $\dot{x}$ being $= b\dot{y} + 2cy\dot{y} + 3dy^2\dot{y} + 4ey^3\dot{y} + \&c.$ we shall have $\dot{u}$ $\left(= \frac{y^2\dot{x}}{2a}\right) = \frac{by^2\dot{y}}{2a} + \frac{2cy^3\dot{y}}{2a} + \frac{3dy^4\dot{y}}{2a} + \frac{4ey^5\dot{y}}{2a} +$ &c. and therefore $u = \frac{by^3}{6a} + \frac{2cy^4}{8a} + \frac{3dy^5}{10a} + \frac{4ey^6}{12a}$ &c. = the true value of the area in this case.

## EXAMPLE XI.

124. *Let it be proposed to find the Area of a Semicircle* A R E H.

Here, putting the diameter $AH = a$, $AB = x$, and $BR = y$, &c. *(as usual)* we have $y^2$ $(BR^2) = ax - x^2$



$(AB \times BH)$, and consequently $\dot{u}$ $(y\dot{x}) = \dot{x}\sqrt{ax - x^2} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} \times \overline{1 - \frac{x}{a}}\Big|^{\frac{1}{2}}$ : which expression not being of the kind described in Art. 83 and 85, that admit of fluents in finite terms, let it therefore be resolved into an infinite series,* and you will have $\dot{u} = a^{\frac{1}{2}} x^{\frac{1}{2}} \dot{x} \times \overline{1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3} - \frac{5x^4}{128a^4}, \&c.} = a^{\frac{1}{2}} \times (x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{2a} - \frac{x^{\frac{5}{2}}\dot{x}}{8a^2} - \frac{x^{\frac{7}{2}}\dot{x}}{16a^3}$ &c.) From whence the fluent of every term being taken, according to the common method, there will come out $u = a^{\frac{1}{2}} \times (\frac{2x^{\frac{3}{2}}}{3} - \frac{x^{\frac{5}{2}}}{5a} - \frac{x^{\frac{7}{2}}}{28a^2} - \frac{x^{\frac{9}{2}}}{72a^3} - \frac{5x^{\frac{11}{2}}}{704a^4}$ &c.) $= x\sqrt{ax} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \frac{x^3}{72a^3} - \frac{5x^4}{704a^4} - \&c.}$ = the area A B R. Now, when $x = \frac{1}{2}a$, the ordinate B R will coincide with the radius O E; in which case the area becomes $= \frac{1}{2}a\sqrt{\frac{1}{2}a^2} \times (\frac{2}{3} - \frac{1}{10} - \frac{1}{112} - \frac{1}{576} - \frac{5}{11264}, \&c.) = \frac{a^2\sqrt{\frac{1}{2}}}{2} \times (0{,}6666 - 0{,}1 - 0{,}0089 - 0{,}0017 - 0{,}0004\ \&c.) = 0{,}1964a^2$; which, multiplied by 2, gives $0{,}3928a^2$ for the area of the semi-circle A E H, nearly.

As the foregoing series, in finding the area of the whole quadrant A O E, converges but slowly, a considerable number of terms ought therefore to be taken to have the conclusion but tolerably exact, the five first terms above collected being sufficient to bring out no more than three places of figures that can be depended on. For which reason it may be of use to consider, whether, by computing the area of some particular portion (A B R) of the said quadrant, that of the whole may not be deduced; where $x$ being small in

* Art. 90 & 99.

comparison of $a$, the series may have such a rate of convergency, that a smaller number of terms will be sufficient.*

Now, in order to this, it is well known that if the arch A R be taken $= \frac{1}{3}$ A E (or 30 degrees) the sine BR will be $= \frac{1}{2}$ AO; and consequently AB ($x$) $=$ AO $-$ O B $=$ A O $- \sqrt{\text{OR}^2 - \text{BR}^2}$; which, if the radius AO be expounded by unity (to facilitate the operation) will be $= 0{,}1339746$ very nearly: this, therefore, with the value of $a$, being substituted in the forementioned series, $\sqrt{ax^3} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - }$ &c. we have $0{,}0693505 \times (0{,}6666666 - 0{,}0133975 - 0{,}0001603 - 0{,}0000042 -$ &c.$) = 0{,}0693505 \times 0{,}6531046 = 0{,}0452931 =$ the area A B R: which, added to the area O B R ($=$ O B $\times \frac{1}{2}$ B R $= \sqrt{\frac{3}{4}} \times \frac{1}{4} = 0{,}2165063$) gives 0,2617994, for the area of the sector A O R; the treble whereof, or 0,7853982 (because A R $= \frac{1}{3}$AE) will therefore be the content of the whole quadrant AOE: which number, found by taking four terms of the series only, is true to the last decimal place.

This conclusion may be otherwise brought out, by finding a series for the other part of the area, included between the radius O E and the ordinate B R: wherein the co-sine OB (instead of the versed sine AB) will be the converging (or variable) quantity.

For, putting O B $= x$, and O R (O A) $= b$, we have $y$ (B R $= \sqrt{\text{OR}^2 - \text{OB}^2} = \overline{b^2 - x^2}|^{\frac{1}{2}}$; and consequently ($y\dot{x}$) the fluxion of the area O B R E† $= \dot{x} \times \overline{b^2 - x^2}|^{\frac{1}{2}} = b\dot{x} - \frac{x^2\dot{x}}{2b} - \frac{x^4\dot{x}}{8b^3} - \frac{x^6\dot{x}}{16b^5} - \frac{5x^8\dot{x}}{128b^7} - \frac{7x^{10}\dot{x}}{256b^9}$ &c. Whence the area itself is $= bx - \frac{x^3}{6b} - \frac{x^5}{40b^3} - \frac{x^7}{112b^5} - \frac{5x^9}{1152b^7} - \frac{7x^{11}}{2816b^9}$ &c.

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* Art. 92.

† Art. 112.

Now, if $x$ (O B) be assumed $= \frac{1}{2}$ A O (so that the arch ER may be $= \frac{1}{3}$ A E) and the value of $b$ (A O) be expounded by unity, we shall have

$$x^3\ \left(= x \times x^2 = ,5 \times \tfrac{1}{4} = \frac{,5}{4}\right) = ,125$$

$$x^5\ \left(= x^3 \times x^2 = \frac{,125}{4}\right) = ,03125$$

$$x^7\ \left(= x^5 \times x^2 = \frac{,03125}{4}\right) = ,0078125$$

$$x^9\ \left(= x^7 \times x^2 = \frac{x^7}{4}\right) = ,0019531+$$

$$x^{11}\ \left(= x^9 \times x^2 = \frac{x^9}{4}\right) = ,0004883-$$

&c.

Which values of the powers of $x$ being respectively divided by 6, 40, 112, 1152, 2816, &c. there will result 0,5000000 − 0,0208333 − 0,0007812 − 0,0000698 − 0,0000085 − 0,0000012 − 0,0000002, &c. = 0,4783057, for the area O B R E in the forementioned circumstance, when $OB = \frac{1}{2} OA$: from which, deducting the triangle OBR $(= \sqrt{\tfrac{3}{4}} \times \tfrac{1}{4} = 0{,}2165063)$ the remainder ,2617994 will consequently be the area of the sector E O R; the treble whereof (because E R is *here* $= \frac{1}{3}$AE) will give the area of the whole quadrant, 0,7853982, *as before.*

## EXAMPLE XII.

125. *Let the Curve, whose Area you would find, be the Cissoid of Diocles; whereof the Equation is* $y^2 = \dfrac{x^3}{a-x}$.

* Art. 112. Here we have $\dot{u}\ (y\dot{x}^*) = \dfrac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a-x}} = \dfrac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}} \times \overline{1-\frac{x}{a}}\Big|^{\frac{1}{2}}}$

$=\frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}}\times\overline{1-\frac{x}{a}}\Big|^{-\frac{1}{2}}$ : which being none of the kind that admit of fluents in finite terms,* let it therefore be resolved into an infinite series, and you will have $\dot{u}=$ $\frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}}\times\overline{1+\frac{x}{2a}+\frac{3x^2}{8a^2}+\frac{5x^3}{16a^3}+\frac{35x^4}{128a^4}+\&c.}=\frac{1}{a^{\frac{1}{2}}}\times$ $\overline{x^{\frac{3}{2}}\dot{x}+\frac{x^{\frac{5}{2}}\dot{x}}{2a}+\frac{3x^{\frac{7}{2}}\dot{x}}{8a^2}+\frac{5x^{\frac{9}{2}}\dot{x}}{16a^3}+\&c.}$ Whence $u$ (the area itself) will come out $=\frac{1}{a^{\frac{1}{2}}}\times\left(\frac{2x^{\frac{5}{2}}}{5}+\frac{x^{\frac{7}{2}}}{7a}+\frac{x^{\frac{9}{2}}}{12a^2}+\frac{5x^{\frac{11}{2}}}{88a^3}+\&c.\right)=x^2\sqrt{\frac{x}{a}}\times\left(\frac{2}{5}+\frac{x}{7a}+\frac{x^2}{12a^2}+\frac{5x^3}{88a^3}+\&c.\right)$

## EXAMPLE XIII.

126. *Let the proposed Curve* CSDR *be of such a Nature, that (supposing* A B *Unity) the Sum of the Areas* CSTBC *and* CDGBC *answering to any two proposed Abscissas* A T *and* A G, *shall be equal to the Area* CRNBC *whose corresponding Abscissa* AN *is equal to,* A T × A G, *the Product of the Measures of the two former Abscissas.*

First, in order to determine the equation of the curve (which must be known before the area can be found) let the ordinates G D and N R move parallel to themselves towards HF; and, then, having put GD $=y$,

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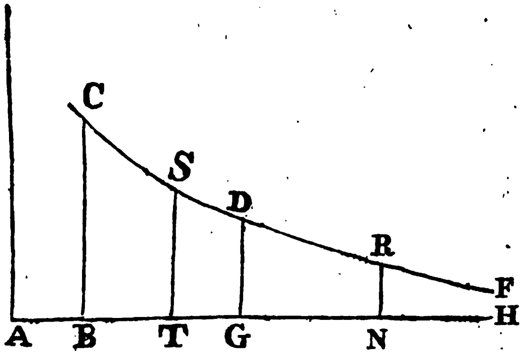
* Art. 83 & 85.

$NR=z$, $AT=a$, $AG=s$, and $AN=u$, the fluxion of the area CDGB will be represented by $y\dot{s}$, and that



of the area C R N B by $z\dot{u}$:* which two expressions must, by the nature of the problem, be equal to each other; because the latter area C R N B, exceeds the former CDGB by the area CSTB, which is here considered as a constant quantity; and it is evident that two expressions, that differ only by a constant quantity, must always have equal fluxions.

[*] Art. 112.

Since, therefore $y\dot{s}$ is $=z\dot{u}$, and $u=as$, *by hypothesis*, it follows that $\dot{u}=a\dot{s}$, and that the first equation (by substituting for $\dot{u}$) will become $y\dot{s}=az\dot{s}$; or $y=az$, or lastly $ys=zas$, that is, $GD\times AG=NR\times AN$: therefore $GD : NR :: AN : AG$; whence it appears that every ordinate of the curve is reciprocally as its corresponding abscissa.

Now, to find the area of the curve so determined, put $BC=b$, and $BG=x$: then, since $AG\ (1+x) : AB\ (1) :: BC\ (b) : GD\ (y)$ we have $y=\frac{b}{1+x}$, and consequently $\dot{u}\ (=y\dot{x})=\frac{b\dot{x}}{1+x}=b\times(\dot{x}-x\dot{x}+x^2\dot{x}-x^3\dot{x}+x^4\dot{x}-\&c.)$ Whence, B G D C, the area itself will be $= b \times (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ &c.) Which was to be found.

It may be here observed that the areas of the spaces above mentioned, are analogous to, and have the very same properties as *logarithms;* and that those spaces, or logarithms, may be of different forms or values, according as you take the value of the first ordinate BC, which may be assumed at pleasure: thus, if B C be taken $=$ AB $=$ unity, the curve will become an equilateral hyperbola whose center is A (because then A G $\times$ GD $=$ AB$^2$) and in that case they are called hyperbolical logarithms: but, if BC be taken $= 0{,}43429448$ (so that the logarithm, or the area of the space CDGB, answering to the abscissa AG, when expressed by the number 10, may be expounded by unity, or AB$^2$) we shall then have the common, or *Briggean* form of logarithms.

From these logarithms (given by the tables) the business of finding fluents, is in many cases, very much facilitated: for, if the fluxion given appears to agree with the fluxion of any known logarithmic expression, its fluent may, it is evident, be had by the tables, ready calculated, without the trouble of an infinite series.

But, now to know what kinds of fluents are explicable by means of logarithms, it will be necessary to observe that, *the fluxion of any hyperbolic logarithm is always expressed by the fluxion of the corresponding number, divided by that number:* this appears from above, where $(y\dot{x})$ the fluxion of the area (or logarithm) BGDC, when BC $=$ AB $= 1$, is truly represented by $\frac{\dot{x}}{1+x}$; where $1 + x$ $(=$ AG$)$ may stand for any number whatever; and $\dot{x}$ for its fluxion.

Hence the fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ will be expressed by the hyperbolical logarithm of $x+\sqrt{x^2\pm a^2}$: for the fluxion of $(x+\sqrt{x^2\pm a^2})$ the number itself, being $\dot{x} + \frac{x\dot{x}}{\sqrt{x^2\pm a^2}} = \frac{\dot{x}\sqrt{x^2\pm a^2}+x\dot{x}}{\sqrt{x^2\pm a^2}} = \frac{\dot{x}}{\sqrt{x^2\pm a^2}} \times \overline{\sqrt{x^2\pm a^2}+x}$, this last quantity, divided by that number, gives $\frac{\dot{x}}{\sqrt{x^2\pm a^2}}$, the very fluxion first proposed.

It also appears that the fluent of $\frac{\dot{x}}{\sqrt{2ax+x^2}}$ will be truly expounded by the hyperbolical logarithm of $a+x+\sqrt{2ax+x^2}$: because the fluxion of the number $(a+x+\sqrt{2ax+x^2})$ is here $= \dot{x} + \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+x^2}} = \frac{\dot{x}}{\sqrt{2ax+x^2}} \times \overline{\sqrt{2ax+x^2}+a+x}$; which divided by that number produces $\frac{\dot{x}}{\sqrt{2ax+x^2}}$.

Likewise the fluent of $\frac{2a\dot{x}}{a^2-x^2}$ will be represented by the hyperbolical logarithm of $\frac{a+x}{a-x}$: because, the fluxion of $\frac{a+x}{a-x}$, being $\frac{\dot{x}\times\overline{a-x}+\dot{x}\times\overline{a+x}}{\overline{a-x}|^2} = \frac{2a\dot{x}}{\overline{a-x}|^2}$, if the same be therefore divided by $\frac{a+x}{a-x}$, we shall have $\frac{2a\dot{x}}{\overline{a-x}|^2}\times\frac{a-x}{a+x} = \frac{2a\dot{x}}{\overline{a-x}\times\overline{a+x}} = \frac{2a\dot{x}}{a^2-x^2}$.

Lastly, the fluent of $\frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$ will be denoted by the hyperbolical logarithm of $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$; for here the fluxion of the number is $\frac{\mp x\dot{x}}{\sqrt{a^2\pm x^2}} \times$

$$\frac{a+\sqrt{a^2\pm x^2}}{\overline{a+\sqrt{a^2\pm x^2}}|^2} \mp \frac{x\dot{x}}{\sqrt{a^2\pm x^2}} \times \frac{a-\sqrt{a^2\pm x^2}}{\overline{a+\sqrt{a^2\pm x^2}}|^2} =$$

$\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2}$; which divided by

$\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$ gives $\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2} \times$

$$\frac{a+\sqrt{a^2\pm x^2}}{a-\sqrt{a^2\pm x^2}} = \frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}\times\overline{a-\sqrt{a^2\pm x^2}}}$$

$= \frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times \mp x^2} = \frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$, the fluxion proposed.

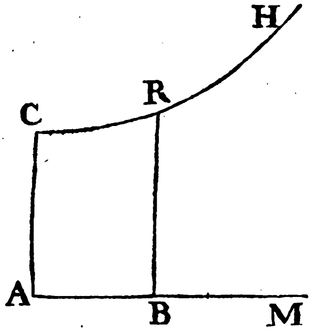
These four are the principal forms of fluxions; whose fluents may be found from a table of logarithms of the hyperbolic kind: which table, upon occasion, may be easily supplied by a table of the common form: for, since the hyperbolical logarithm of any number is to the common logarithm of the same number, in the constant ratio of unity to 0,43429448 (as appears from above) it follows that if any common logarithm be, either, divided by 0,43429448, or multiplied by its reciprocal 2,30258509, you will thence obtain the hyperbolical logarithm corresponding.

## EXAMPLE XIV.

127. *Let it be required to determine the Area of the Curve; whose Equation is* $a^2y - x^2y - a^3 = 0$.

* Art. 112. In which case $y$ being $= \frac{a^3}{a^2 - x^2}$, we have $\dot{u}$ $(= y\dot{x})$\*

$$= \frac{a^3\dot{x}}{a^2 - x^2} = a\dot{x} + \frac{x^2\dot{x}}{a} + \frac{x^4\dot{x}}{a^3} + \frac{x^6\dot{x}}{a^5} + \frac{x^8\dot{x}}{a^7} + \&c.$$



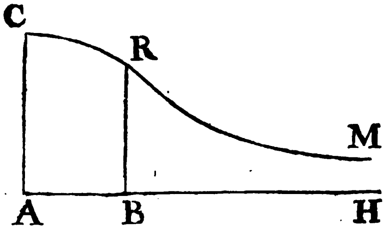
Whence $u = ax + \frac{x^3}{3a} + \frac{x^5}{5a^3} + \frac{x^7}{7a^5} + \frac{x^9}{9a^7} + \&c.$ = the area sought.

But the same area (or fluent) may be found without an infinite series, by means of a table of logarithms, agreeable to the observations in the last article: for, since it there appears that the fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ is truly expressed by the hyperbolic logarithm of $\frac{a+x}{a-x}$, it follows that that of $\frac{a^3\dot{x}}{a^2 - x^2}$ $(= \frac{2a\dot{x}}{a^2 - x^2} \times \frac{1}{2}a^2)$ will be expressed by the same logarithm multiplied by $\frac{1}{2}a^2$. Thus, for example sake, let $a$ $( = AC)$ be taken $=10$, and $x$ $(=\text{AB})=5$; then will $\frac{a+x}{a-x}=3$; whose logarithm taken from the common tables is 0,4771213; which multiplied by the *modulus* 2,30258509 (see the last article) gives 1,09861228 for the hyperbolical logarithm of $\frac{a+x}{a-x}$; and this again multiplied by 50 $(\frac{1}{2}a^2)$ produces 54,930614 for the true value of the area A B R C, in the aforesaid circumstance, when A C $=10$, and A B $=5$.

## EXAMPLE XV.

128. *Where the proposed Curve is that whose Equation is* $a^2y^2+x^2y^2=a^4$.

Here, by reducing the given equation, we get $y=\frac{a^2}{\sqrt{a^2+x^2}}$: therefore $y\dot{x}=\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}=\dot{u}$.* Whence the fluent of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ being $=$ hyperbolical



log. of $x+\sqrt{a^2+x^2}$ (by Art. 126) that of $\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}$ will consequently be $=$ the same logarithm multiplied by $a^2$.

But to find whether the fluent thus determined does not need a correction,† let $x$ be taken $=0$; then the

* Art. 112.

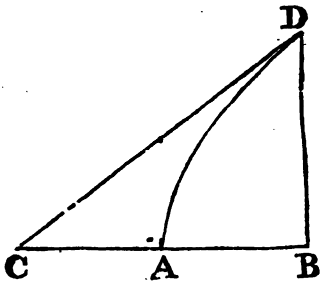
† Art. 78.

fluent will become = hyp. log. $a \times a^2$: which, therefore, must be subtracted, to have the true value of the
* Art. 78. area A C R B;* and then there results $a^2 \times$ hyp. log. $(x + \sqrt{a^2 + x^2}) - a^2 \times$ hyp. log. $a = a^2 \times$ hyp. log. $\dfrac{x + \sqrt{a^2 + x^2}}{a} = u$.

## EXAMPLE XVI.

129. *Let it be proposed to find the Area of the Hyperbola* A B D, *and also the Area of the hyperbolical Sector* CAD; *supposing* C *to be the Center, and* A *the principal Vertex of the Curve.*

Here, putting the semi-transverse axis C A $= a$, the semi-conjugate $= c$, and C B $= x$; we have, by the



property of the curve, $y$ (= B D) $= \dfrac{c}{a}\sqrt{x^2 - a^2}$ and therefore $\dot{u} = y\dot{x} = \dfrac{c\dot{x}}{a}\sqrt{x^2 - a^2}$ = the fluxion
† Art. 112. of the area A B D.†

But to find the fluxion of the sector C A D, it is to be observed, that as the said sector is = C B D − A B D $= \dfrac{xy}{2} - u$, its fluxion will therefore be =

$\frac{x\dot{y}}{2}+\frac{y\dot{x}}{2}-\dot{u}=\frac{x\dot{y}}{2}-\frac{y\dot{x}}{2}$ (because $\dot{u}=y\dot{x}$)* which, by substituting for $y$ and $\dot{y}$, their equals $\frac{c}{a}\sqrt{x^2-a^2}$ and $\frac{cx\dot{x}}{a\sqrt{x^2-a^2}}$, is at length reduced to $\frac{ac}{2}\times\frac{\dot{x}}{\sqrt{x^2-a^2}}$: whereof the fluent (by Art. 126) is $\frac{ac}{2}$ × hyp. log. $x+\sqrt{x^2-a^2}$; which corrected (by making $x=a$) will become $\frac{ac}{2}$ × hyp. log. $(x+\sqrt{x^2-a^2})-\frac{ac}{2}$ × hyp. log. $a=\frac{ac}{2}$ × hyp. log. $\frac{x+\sqrt{x^2-a^2}}{a}$ = the sector A D C: which, subtracted from $\frac{cx\sqrt{x^2-a^2}}{2a}$ $(=\frac{\mathrm{BC}\times\mathrm{BD}}{2}$ = the triangle A B D) leaves $\frac{cx\sqrt{x^2-a^2}}{2a}-\frac{ac}{2}$ × hyp. log. $\frac{x+\sqrt{x^2-a^2}}{a}$ for the required area of the hyperbola A B D.

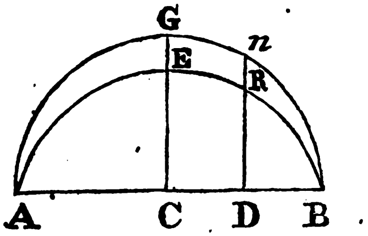
## EXAMPLE XVII.

130. *Let the Curve proposed be the Ellipsis* A E B.

Then, putting the transverse axis A B = $a$, and the conjugate (2 C E) = $c$; we shall, by the property of the curve, have $y$ (D R) = $\frac{c}{a}\sqrt{ax-x^2}$, and therefore $\dot{u}$ $(y\dot{x})=\frac{c}{a}\times\dot{x}\sqrt{ax-x^2}$ = the fluxion of the area A R D.

* Art. 112.

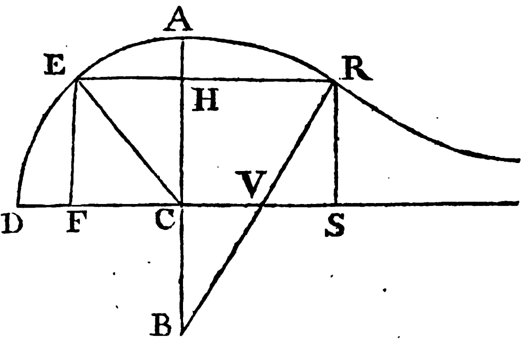
But $\dot{x}\sqrt{ax-x^2}$ is known to express the fluxion of the corresponding segment A D $n$ of the circumscribing semi-circle; whose fluent is, therefore, given, by Art. 124; which being denoted by A, that of $\frac{c}{a} \times \dot{x}\sqrt{ax-x^2}$ will, consequently, be $= \frac{c}{a} \times$ A. Hence, the area of the segment of an ellipsis, is to the area of the corresponding segment of its circumscribing circle, as the lesser axis of the ellipsis is to the greater; whence, it follows that the whole ellipsis must be to the whole circle in the same ratio.



## EXAMPLE XVIII.

131. *Let the Curve* A R &c. *whose Area* C A R S *you would find, be the Conchoid of* Nicomedes.

Whereof the equation (putting B C$=a$, and R V $(=\text{AC} = b)$ is $x^2y^2 = \overline{a+y}|^2 \times \overline{b^2-y^2}$ (*Vide* Art. 57). Which, by reduction, becomes $x = \frac{a\sqrt{b^2-y^2}}{y} +$



$\sqrt{b^2 - y^2}$: but, to bring it down to a, *still*, more simple form, make $\sqrt{b^2 - y^2}$ ($=SV$) $= z$; then $y = \sqrt{b^2 - z^2}$; whence, by substitution, $x = \frac{az}{\sqrt{b^2 - z^2}} + z$; and consequently $\dot{x} =$

$$\frac{a\dot{z}\sqrt{b^2 - z^2} + \frac{z\dot{z}}{\sqrt{b^2 - z^2}} \times az}{b^2 - z^2} + \dot{z} =$$

$$\frac{a\dot{z} \times \overline{b^2 - z^2} + az^2\dot{z}}{\overline{b^2 - z^2} \times \sqrt{b^2 - z^2}} + \dot{z} = \frac{ab^2\dot{z}}{\overline{b^2 - z^2} \times \sqrt{b^2 - z^2}} + \dot{z};$$

and therefore $\dot{u}$ $(y\dot{x}) = \sqrt{b^2 - z^2} \times \left(\frac{ab^2\dot{z}}{\overline{b^2 - z^2} \times \sqrt{b^2 - z^2}} + \dot{z}\right) = \frac{ab^2\dot{z}}{b^2 - z^2} + \dot{z}\sqrt{b^2 - z^2}$.

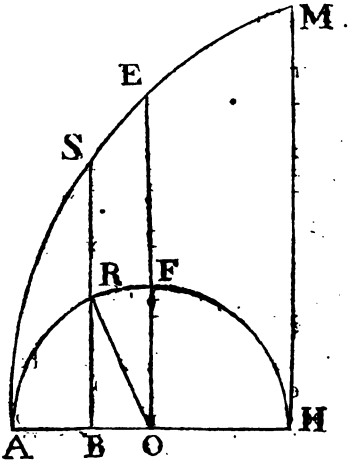
But now, to exhibit the fluent hereof; upon C, as a center, with the radius A C *(b)* let a quadrant of a circle AED be described, and let R H, produced, meet the periphery thereof in E, also let E F be parallel to A C, and let C E be drawn: it is evident (because C E (C A) = V R and E F = R S) that C F is also = V S $= z$; and therefore, E F being ($= \sqrt{CE^2 - CF^2}$) $= \sqrt{b^2 - z^2}$, it appears that $\dot{z}\sqrt{b^2 - z^2}$ (the second

term of our given quantity) expresses the fluxion of the area A E F C: whence, if to this area (found *by the table of Segments)* the fluent of the first term
* Art. 126. $\frac{ab^2\dot{z}}{b^2 - z^2}$, or the hyp. log. of $\frac{b+z}{b-z}$, $\times \frac{1}{2} ab$,* be added, the sum will be the whole area A R C S, that was to be determined.

## EXAMPLE XIX.

132. *Let it be required to determine the Area* A S R A *included by the common Cycloid* A S M *and its generating Semi-circle* A R H.

Put the radius A O (or R O) $= a$, the sine B R $= y$, the co-sine O B $= x$, and the arch A R ($=$ R S, by the property of the cycloid) $= z$: then A B being $= a$



$- x$ its fluxion will be $- \dot{x}$; whence ($\dot{w}$) that of the
† Art. 112. area ARS is $= - z\dot{x}$.† Now to find the fluent thereof, make $w = - zx$ ($=$ the fluent, if $z$ was con-

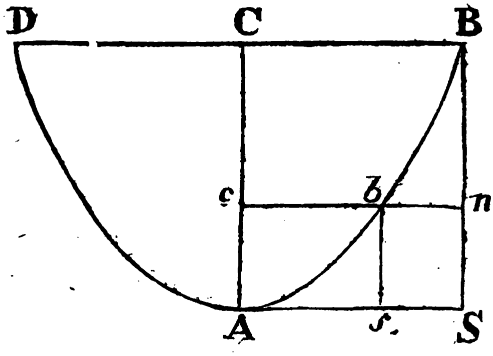
stant) then $\dot{w}$ being $= -z\dot{x} - x\dot{z}$,* we shall have $\dot{u}$ $(= -z\dot{x})$ $= \dot{w} + x\dot{z}$. But (by Art. 35) $\dot{z}$ (A R fluxion) : $\dot{y}$ (B R fluxion) :: radius : co-sine of the angle ARB, or its equal ROB :: OR $(a)$ : OB $(x)$: therefore, by multiplying extremes and means, we get $x\dot{z} = a\dot{y}$: whence, by substitution $\dot{u}$ $(= \dot{w} + x\dot{z}) = \dot{w} + a\dot{y}$; and consequently, by taking the fluent, $u = w + ay = -zx + ay =$ AO × BR − BO × AR = the area A R S.

Hence it follows that the area (A E F A) when R B coincides with the radius F O, is barely = A O × F O = A O$^2$: and that the whole area A M H F A is truly defined by −ARH × −OH, or by ARH × OH; that is by four times the area of the generating semi-circle.

## EXAMPLE XX.

133. *Let the Curve proposed be the* Catenaria D A B.

Then, drawing B S and $bs$ parallel to the axis A C, and AS and $cbn$ perpendicular to the same; and making (as usual) A$c = x$, $cb = y$ and A$b = z$, we shall have, by



the property of the curve, $2ax + x^2 = z^2$: whence $x = \sqrt{a^2 + z^2} - a$, and $\dot{x} = \dfrac{z\dot{z}}{\sqrt{a^2 + z^2}}$: from which the

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[*] Art. 10.

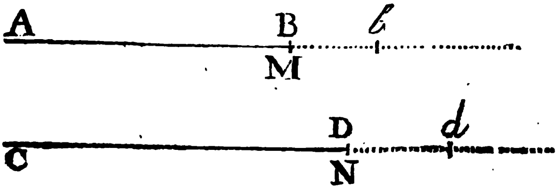
* Art. 135. value of $\dot{y}$ (which in all curves is $= \sqrt{\dot{z}^2 - \dot{x}^2}$)* will here be found $= \sqrt{\dot{z}^2 - \frac{z^2\dot{z}^2}{a^2+z^2}} = \sqrt{\frac{a^2\dot{z}^2}{a^2+z^2}} \times \frac{a\dot{z}}{\sqrt{a^2+z^2}}$; and this multiplied by $\sqrt{a^2+z^2} - a$ $(= bs)$ gives $a\dot{z} - \frac{a^2\dot{z}}{\sqrt{a^2+z^2}}$ (= the rectangle S$b$)
† Art. 112. = the fluxion of the area A$sb$.† From whence, by taking the fluent, the area itself is found $= az - a^2$
‡ Art. 126. $\times$ hyp. log. $\frac{z+\sqrt{a^2+z^2}}{a}$:‡ which therefore deducted from the rectangle $sc$ $(= yx = y\sqrt{a^2+z^2} - ay)$ leaves $y\sqrt{a^2+z^2} - ay - az, + a^2 \times$ hyp. log. $\frac{z+\sqrt{a^2+z^2}}{a}$ for the required area A$bc$. But, since $\dot{y} = \frac{a\dot{z}}{\sqrt{a^2+z^2}}$ we have $y = a \times$ hyp. log. $\frac{z+\sqrt{a^2+z^2}}{a}$; whence, by substitution, the area, at last comes out $= y\sqrt{a^2+z^2} - az$, or $= a\sqrt{a^2+z^2} \times$ hyp. log. $\frac{z+\sqrt{a^2+z^2}}{a} - az$.

## SCHOLIUM.

134. At the beginning of this, and in the preceding sections, we have seen how the fluxions of quantities are determined, *by conceiving the generating motion to become uniform at the proposed position;* according to the
§ Art. 2. true definition of a fluxion: § but hitherto no particular notice has been taken of *the method of increments, or indefinitely little parts,* used (and mistaken) by *many* for *that* of fluxions: in which the operations are, for the general part, exactly the same; and which (though less accurate) may be applied to good purpose in finding the fluxions themselves, in many cases. For which reasons it may not be improper to add here

a few lines on that head, to show the beginner how the two methods differ from each other; especially as we shall be enabled, from thence, to draw out some conclusions that will be of use in the ensuing part of the work.

It hath been frequently inculcated in the foregoing pages, that *the fluxions of quantities are always measured by how much the quantities themselves* would be *uniformly augmented in a given time.* Therefore, if two



quantities or lines, A B and C D be generated together, by the uniform (or equable) motion of two points B and D, it follows, that any two spaces B *b* and D *d* *actually* gone over (whereby A B and C D are augmented) in the same time, will truly express the fluxions of the generated lines A B and C D: whence it appears that the increments (or spaces actually gone over) and the fluxions are the same in this case, where the generating velocities are equable.

But if, on the contrary, the velocities of the two points, in generating the increments M *b* and N *d*, be supposed either to increase, or to decrease, the lines or increments so generated will, it is plain, no longer express the fluxions of A B and C D; being greater, or less than the spaces that *might be uniformly* described, in the same time, with the velocities at M and N.

If, indeed, those increments, and the time of their description, be taken so exceeding small that the motion of the points during that time may be considered as equable, the ratio of the said increments will then express that of the fluxions, or be as the velocity at M to that at N, indefinitely near; but cannot be conceived to be *strictly so;* unless, perhaps, in certain particular cases.

Hence we see that the *differential method*, which proceeds upon these indefinitely little increments (actually generated) as we do upon fluxions (or the spaces that *might be uniformly* generated) differs little, or nothing, from the method of fluxions, except in the manner of conception, and in point of accuracy, wherein it appears defective: and yet it is very certain the conclusions this way derived are *mathematically* true; which has afforded matter of wonder to *some:* but the reason why they are so is very easily explained. For, although the *whole complete* increment is actually understood by the notation and first definition (of this method) yet in the solution of problems the exact measure thereof is not taken, but only that part of it which would arise from an uniform increase, agreeable to the notion of a fluxion; which admits of a strict demonstration: but, after all, the *differential method* has one advantage above that of fluxions, which is, we are not there obliged to introduce the properties of motion. Since we reason upon the increments themselves, and not upon the manner in which they may be generated.

It has been hinted above, that, though the increments of quantities are not, *strictly*, as the fluxions, yet from them the ratio of the fluxions may be deduced; and it appears that the smaller those increments are taken, the nearer their ratio will approach to that of the fluxions. Therefore, if we can by any means, find the ratio to which the said increments, by conceiving them less and less, do perpetually converge, and which they may approach, before they vanish, nearer than by any assignable difference, that ratio (called hereafter, for distinction sake, *the ratio limiting that of the increments)* will be, *strictly*, that of the fluxions.

This will more particularly appear from the following instances; wherein the manner of deriving the ratio of the fluxions, from that of the increments, is shown.

### 1°. *Let it be proposed to determine the Ratio of the Fluxions of $x$ and $x^2$.*

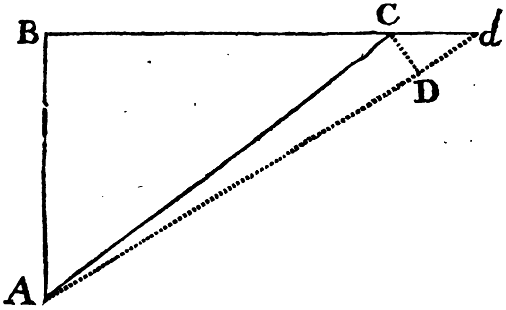
Now, if $x$ be supposed to be augmented by any (small) quantity $\acute{x}$, so as to become $x+\acute{x}$, its square ($x^2$) will be augmented to $(x+\acute{x})^2 = x^2 + 2x\acute{x} + \acute{x}^2$; whence the increment of $x^2$ will be $2x\acute{x}+\acute{x}^2$; which therefore is to ($\acute{x}$) the increment of $x$, as $2x+\acute{x}$ to 1. Hence, because the lesser $\acute{x}$ is taken, the nearer this ratio approaches to that of $2x$ to 1, which is its *limit*, the ratio of the fluxions will therefore be expressed by that of $2x$ to 1, or, which is the same, by that of $2x\dot{x}$ to $\dot{x}$ *(as in Art.* 6.*)*

### 2°. *Let the Ratio of the Fluxions of $x$ and $x^n$ be required.*

Then, if $x$ be augmented to $x+\acute{x}$, $x^n$ will be augmented to $\overline{x+\acute{x}}|^n = x^n + nx^{n-1}\acute{x} + \frac{n}{1}\times\frac{n-1}{2}\times x^{n-2}\acute{x}^2 + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}x^{n-3}\acute{x}^3$, &c. (*Vide Art.* 99.) Whence the increments of $x$ and $x^n$ will be to each other as 1 to $nx^{n-1} + \frac{n}{1}\times\frac{n-1}{2}x^{n-2}\acute{x} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}x^{n-3}\acute{x}^2$, &c. Where the smaller $\acute{x}$ taken, the nearer the ratio will approach to that of

1 to $nx^{n-1}$; which appears to be its limit: therefore this last ratio, or that of $\dot{x}$ to $nx^{n-1}\dot{x}$, is the ratio of the fluxions required. (*Vide Art.* 8.)

3°. *Let it be proposed to determine the Proportion of the Fluxions of the Sides* A C *and* B C, *of a right-angled, plane Triangle* A B C; *supposing the Perpendicular* A B *to remain invariable.*



If C $d$ be assumed to represent any increment of BC; and D$d$, the corresponding increment of A C (= A D) the ratio of those increments will be universally expressed by that of the sine of the angle C D $d$ to the sine of the angle D C $d$ *(by plane trigonometry)* and the less the increments are supposed to be, the nearer will the angle C D $d$ approach to a right one, or to an equality with B, which is its limit: and the nearer will D C $d$ approach, at the same time, to an equality with B A C. Therefore *the ratio here limiting that of the increments* is that of the sine of B (or radius) to the sine of B A C: which also expresses that of the required fluxions. (*Vide Art.* 35.)

In the same way the proportion of the fluxions of other kinds of algebraical and geometrical quantities may be investigated; but it will be unnecessary to dwell longer upon this head: I shall therefore only add one other observation from hence (which will be of use hereafter) relating to the value of an algebraic fraction, in that particular circumstance when both its numerator and denominator become equal to nothing, or vanish, at the same time. Which value (it follows from above) will *be found by dividing the fluxion of the numerator by that of the denominator.*

For, since the value of any fraction, in that circumstance, is to be looked on as *the limiting ratio* towards which its two terms converge, before they vanish, and seeing the fluxions are always expressed by that ratio, the truth of the rule, or position, is manifest.

An example, however, may not be improper:

Let, therefore, the fraction $\frac{x^2-a^2}{x-a}$ be propounded, to find the value thereof when $x=a$. In which case, the true value sought, or the fluxion of the numerator divided by that of the denominator, is $=\frac{2x\dot{x}}{\dot{x}}$ $=2x=2a$. And that this is the true value, may be confirmed by common division, whereby the fraction proposed is reduced to $x+a$; whose value, when $x=a$, is therefore $=2a$, *the very same as before.*

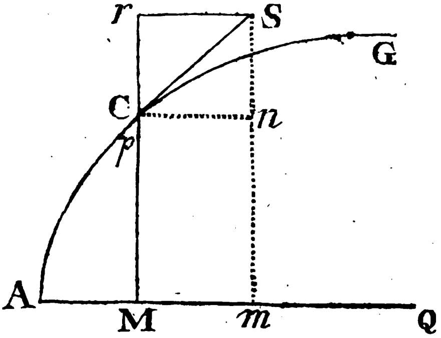
# SECTION VIII.

## *The Use of Fluxions in the Rectification, or finding the Lengths, of Curves.*

### CASE I.

135. *LET* A C G *be a Curve of any kind whose Ordinates are parallel to themselves, and perpendicular to the Axis* A Q.

If the fluxion of the abscissa A M be denoted b M *m*; or by C *n* (equal and parallel to M *m*) and *n* S



equal and parallel to $\dot{C}$ *r*, be taken to represent the corresponding fluxion of the ordinate M C; then will the diagonal C S (touching the curve in C*) be the line which the generating point (*p*) would describe, was its motion to become uniform at C (*Vide* Art. 48 and 49), which line, therefore, truly expresses the fluxion of the space A C gone over, according to *the definition.*†

Hence, putting A M $=x$, C M $=y$, and A C $=z$, we have $\dot{z}$ $(= CS = \sqrt{Cn^2 + Sn^2}) = \sqrt{\dot{x}^2 + \dot{y}^2}$; from which, and the equation of the curve, the value of $z$ may be determined.

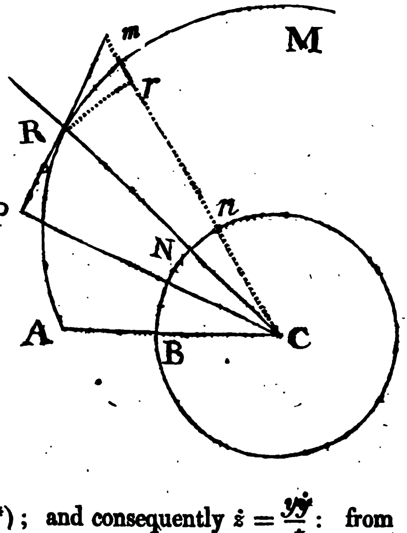
* Art. 48 & 49.

† Art. 2.

## CASE II.

136. *Let all the Ordinates of the proposed Curve* A R M *be referred to a Center* C.

Then, putting the tangent R P (intercepted by the perpendicular C P) $=t$, the arch B N, of a circle described about the center C $=x$; the radius C N (or C B) $=a$, &c. (*Vide* Art. 113) we have $\dot{z} : \dot{y} :: y$ (C R)



$: t$ (R P*); and consequently $\dot{z} = \frac{y\dot{y}}{t}$: from whence the value of $z$ will be found, if the relation of $y$ and $t$ is given.

But in other cases it will be better to work from the following equation, viz. $\dot{z} = \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{a^2}}$. Which is thus derived.

Let the right-line C R be conceived to revolve about the center C; then since the celerity of the gene-

* Art. 35.

nerating point R in a direction perpendicular to C R is to ($\dot{x}$) the celerity of the point N, as C R ($y$) to C N ($a$) it will therefore be truly represented by $\frac{y\dot{x}}{a}$: which being to ($\dot{y}$) the celerity in the direction of C R, produced, as C P ($s$) : R P ($t$)* it follows that $\frac{y^2\dot{x}^2}{a^2} : \dot{y}^2 :: s^2 : t^2$: whence, by composition, $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 : \dot{y}^2 :: s^2 + t^2 \; (y^2) : t^2$; therefore $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 = \frac{y^2\dot{y}^2}{t^2}$, and consequently $\sqrt{\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2} \; \left(= \frac{y\dot{y}}{t}\right) = \dot{z}$; as was to be shown.

But the same conclusion may be more easily deduced from the increments of the flowing quantities, according to the preceding scholium.

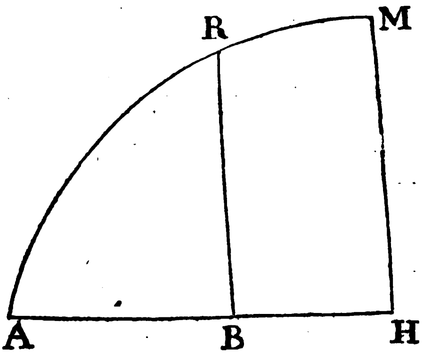
For, if R$m$, $rm$, and N$n$ be assumed to represent ($\acute{z}$, $\acute{y}$ and $\acute{x}$) any very small corresponding increments of A R, C R, and B N, it will be as C N ($a$) : C R ($y$) :: $\acute{x}$ (the arch N$n$) : the similar arch R$r$ $= \frac{y\acute{x}}{a}$. And, if the triangle R$rm$ (which, while the point $m$ is returning back to R, approaches continually nearer and nearer to a similitude with C R P) be considered as *rectilineal*, we shall also obtain $\acute{z}^2$ ($=$ R$m^2 =$ R$r^2 + rm^2$) $= \frac{y^2\acute{x}^2}{a^2} + \acute{y}^2$: whence, by writing $\dot{z}$, $\dot{x}$, and $\dot{y}$ for $\acute{z}$, $\acute{x}$, and $\acute{y}$ (according to the scholium) there comes out $\dot{z}^2 = \frac{y^2\dot{x}^2}{a^2} + \dot{y}^2$, *as before.*

* Art. 35.

## EXAMPLE I.

137. *Let the Curve* A R M, *whose Length is sought, be the Semi-cubical Parabola.*

Whereof the equation being $ax^2 = y^3$, or $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ we thence have $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$ : whence $\dot{z}$ $(= \sqrt{\dot{y}^2 + \dot{x}^2}$ *$)$



$= \sqrt{\dot{y}^2 + \frac{9y\dot{y}^2}{4a}} = \frac{\dot{y} \times \overline{4a + 9y}|^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$. Whose fluent (found by the common rule) is $\frac{\overline{4a + 9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$; which, corrected (by making $y = 0$) becomes $\frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27} = z$

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* Art. 135.

## EXAMPLE II.

138. *Let the Curve proposed be a Parabola of any (other) kind.*

Then $x = \frac{y^n}{a^{n-1}}$ being a general equation to all kinds of parabolas, we here have $\dot{x} = \frac{ny^{n-1}\dot{y}}{a^{n-1}}$, and therefore $\dot{z}$ $( = \sqrt{\dot{y}^2 + \dot{x}^2}) = \sqrt{\dot{y}^2 + \frac{n^2y^{2n-2}\dot{y}^2}{a^{2n-2}}} =$ $\dot{y} \times \overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}}$: whose fluent, universally expressed in an infinite series, is $y + \frac{n^2y^{2n-1}}{\overline{2n-1} \times 2a^{2n-2}} - \frac{n^4y^{4n-3}}{\overline{4n-3} \times 8a^{4n-4}} + \frac{n^6y^{6n-5}}{\overline{6n-5} \times 16a^{6n-6}}$, &c. $= z$.

But, when $2n-2$, the index of $y$, in the given fluxion, is either equal to unity, or to any *aliquot part* of it, the fluent may be accurately had in finite terms, by Art. 84.

For, by putting $\frac{1}{2n-2} = v$, and $\frac{n^2}{a^{2n-2}} = c$, our fluxion $(\overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}} \times \dot{y})$ is, in the first place, reduced to $\overline{1 + cy^{\frac{1}{v}}}\Big|^{\frac{1}{2}} \times \dot{y}$: which, being compared with $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, the general expression in the aforesaid *article*, we have $a=1$, $z=y$, $n=\frac{1}{v}$ $m=\frac{1}{2}$, $d=1$, $\dot{z}=\dot{y}$, $rn-1=0$, or $\frac{r}{v}-1=0$; whence $r=v$, $s\ (r+m)=v+\frac{1}{2}$; and consequently

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\&c.}^{*}$$

$$\frac{\overline{1+cy^{\frac{1}{v}}}|^{\frac{3}{2}}}{c+\frac{c}{2v}}\times\left(y^{\frac{v-1}{v}}-\frac{\overline{v-1}\times y^{\frac{v-2}{v}}}{\overline{v-\frac{1}{2}}\times c}+\right.$$

$$\left.\frac{\overline{v-1}\times\overline{v-2}\times y^{\frac{v-3}{v}}}{\overline{v-\frac{1}{2}}\times\overline{v-\frac{3}{2}}\times c^2}-\&c.\right)=\text{the fluent of}$$

$\overline{1+cy^{\frac{1}{v}}}|^{\frac{1}{2}}\times\dot{y}$; which was to be determined, and which will (it is plain) always terminate in $v$ terms, when $v$, or its equal $\frac{1}{2n-2}$, is a whole positive number.

If $\frac{2v+1}{2v}$ (derived from $v=\frac{1}{2n-2}$) be substituted for its equal $n$, the equation of the curve, will be changed to $ax^{2v}=y^{2v+1}$; which, if $v$ be expounded by 1, 2, 3, 4, &c. successively, will become $ax^2=y^3$, $ax^4=y^5$, $ax^6=y^7$, $ax^8=y^9$, &c. respectively: in all which cases the length of the curve may therefore be *accurately* had from the fluent above exhibited.

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* Art. 24.

Moreover, if $n$ be assumed $= 2$ (or $v = \frac{1}{2}$) the general equation, $x = \frac{y^n}{a^{n-1}}$, will then become $x = \frac{y^2}{a}$; answering to the common (or conical) parabola. And therefore in that case $\dot{z}$ $\left( = \overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}} \times \dot{y} \right)$ is $= \dot{y}\sqrt{1 + \frac{4y^2}{a^2}} = \frac{\dot{y}\sqrt{\frac{1}{4}a^2 + y^2}}{\frac{1}{2}a} = \frac{\dot{y}\sqrt{b^2 + y^2}}{b}$ (by putting $b = \frac{1}{2}a$) $= \frac{\dot{y} \times \overline{b^2 + y^2}}{b\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2\dot{y} + y^2\dot{y}}{\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2y\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2\dot{y}}{\sqrt{b^2 + y^2}}$: where, the fluent of the first term (of the fluxion so transformed) being $= \frac{1}{2}\sqrt{b^2y^2 + y^4}$ (or $\frac{1}{2}y\sqrt{b^2 + y^2}$) by the common rule; and that of the second term $= \frac{1}{2}b^2 \times$ hyp. log. $\frac{y + \sqrt{b^2 + y^2}}{b}$,* it follows that the length of the curve will, in this case, be $= \frac{\frac{1}{2}y\sqrt{b^2 + y^2}}{b} + \frac{1}{2}b \times$ hyp. log. $\frac{y + \sqrt{b^2 + y^2}}{b}$

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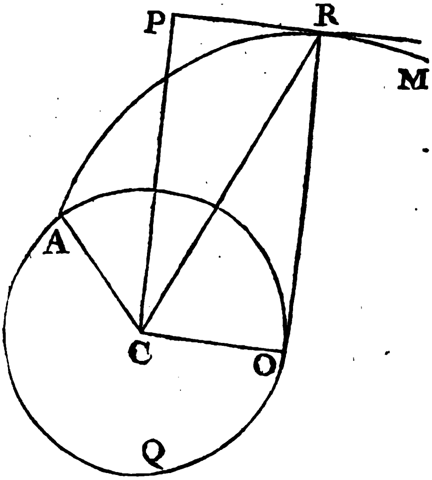
* Art. 126.

## EXAMPLE III.

139. *Let the Curve proposed be the Involute of a Circle;* whose nature is such, that the part P R of the tangent intercepted by the point of contact and the perpendicular C P, is every where equal to the radius C O of the generating circle: therefore $\dot{z}$ $\left(=\frac{y\dot{y}}{t}\,*\right)$ being here $=\frac{y\dot{y}}{a}$, we first get $z=\frac{y^2}{2a}$, which corrected, by making $y=a$ (= A C) becomes $\frac{y^2-a^2}{2a}$ $\left(\frac{CP^2}{2CA}\right)$ the true measure of the required arch A R.



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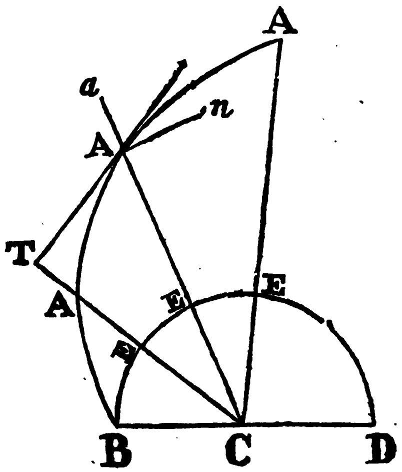
* Art. 136.

## EXAMPLE IV.

140. *In which the Spiral of* Archimedes *is proposed.*

Where the value of $t$ (AT) being denoted by $\frac{by}{\sqrt{b^2+y^2}}$ (*Vide* Art. 62) we get $\dot{z}$ $\left(=\frac{y\dot{y}}{t}\right)$ $=\frac{\dot{y}\sqrt{b^2+y^2}}{b}$: which fluxion being exactly the



same as that expressing the arch of the common parabola, found in Article 138, its fluent will therefore be truly represented by the measure of the said arch, or by $\frac{\frac{1}{2}y\sqrt{b^2+y^2}}{b} + \frac{1}{2}b \times$ hyp. log. $\frac{y+\sqrt{b^2+y^2}}{b}$, the value there exhibited.

## EXAMPLE V.

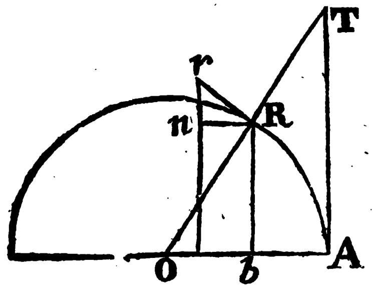
141. *Let the Curve be a spiral whose Equation is* $a^{m-1}x=y^m$ (*Vide* Art. 136.)

In which case $\dot{x}$ being $=\frac{m\dot{y}y^{m-1}}{a^{m-1}}$, it is evident that $\dot{z}$ $\left(=\sqrt{\dot{y}^2+\frac{y^2\dot{x}^2}{a^2}}\right)$* $=\sqrt{\dot{y}^2+\frac{m^2y^{2m}\dot{y}^2}{a^{2m}}}$ * Art. 136. $=\dot{y}\sqrt{1+\frac{m^2y^{2m}}{a^{2m}}}$; and therefore $z=y+\frac{m^2y^{2m+1}}{\overline{2m+1}\times 2a^{2m}}-\frac{m^4y^{4m+1}}{\overline{4m+1}\times 8a^{4m}}+\frac{m^6y^{6m+1}}{\overline{6m+1}\times 16a^{6m}}$ &c. Which value may be otherwise had, without an infinite series, when $\frac{1}{2m}$ is a whole positive number, *Vide* Art. 138.

## EXAMPLE VI.

142. *Where, the Right-sine, Versed-sine, Tangent, or Secant of an Arch of a Circle, being given, it is required to find the length of the Arch itself in terms thereof.*

Put the versed-sine A $b=x$, the right-sine R $b=y$, the tangent A T $=t$, the secant O T $=s$, the arch A R $=z$, and the radius A O, or R O, $=a$; also let R$n=\dot{x}$, $n$ $r$ $=\dot{y}$ and R $r=\dot{z}$: since the angle $r$ $n$ R ( = right-angle) = O $b$ R, and $r$ R $n$ ( = right-angle $-$ $n$ R O) = O R $b$, the triangles $r$ R $n$ and O R $b$



are therefore equi-angular; and it will be, R*b* $(y)$ : OR $(a)$ :: R*n* $(x)$ : R*r* $(\dot{z}) = \frac{a\dot{x}}{y} = \frac{a\dot{x}}{\sqrt{2ax-x^2}}$ (because, by the property of the circle $\sqrt{2ax-x^2}=y$). Also, O*b* $(\sqrt{a^2-y^2})$ : OR $(a)$ :: *nr* $(\dot{y})$ R*r* $(\dot{z}) = \frac{a\dot{y}}{\sqrt{a^2-y^2}}$. These two values exhibit the fluxion of the arch in terms of the versed-sine and right-sine respectively; but, to get the same, in terms of the tangent and secant, we have (*by sim. triangles*) OT $(=s=\sqrt{a^2+t^2})$ : OA $(a)$ :: OR $(a)$ : O$b = \frac{a^2}{s} = \frac{a^2}{\sqrt{a^2+t^2}}$: hence A$b = a - \frac{a^2}{s} = a - \frac{a^2}{\sqrt{a^2+t^2}}$; whose fluxion is therefore $= \frac{a^2\dot{s}}{s^2} = \frac{a^2t\dot{t}}{\overline{a^2+t^2}|^{\frac{3}{2}}}$: whence (again by similar triangles) AT $(=\sqrt{s^2-a^2}=t)$ : OT $(=s=\sqrt{a^2+t^2})$ :: R*n* : R*r* $= \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}} = \frac{a^2\dot{t}}{a^2+t^2} = \dot{z}$.

Now, from any one of the four forms of fluxions $\left(\frac{a\dot{x}}{\sqrt{2ax-x^2}}, \frac{a\dot{y}}{\sqrt{a^2-y^2}}, \frac{a^2\dot{t}}{a^2+t^2}, \frac{a^2\dot{s}}{s\sqrt{s^2-a^2}}\right)$ here found, the value of the arch itself (by taking the fluent, in an infinite series), will likewise become known.

But the third form, expressed in terms of the tangent, being entirely free from radical quantities, will be the most ready in practice, especially where the required arch is but small; though the series arising from the first form, always converges the fastest.

If, therefore, $\frac{a^2\dot{t}}{a^2 + t^2}$ be now converted to an infinite series, we shall have $\dot{z} = \dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6}$ &c. and consequently $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} + \frac{t^9}{9a^8}$ &c. = A R. Where, if *(for example's sake)* A R be supposed an arch of 30 degrees, and A O (to render the operation more easy) be put = unity, we shall have $t = \sqrt{\frac{1}{3}} = .5773502$ (because O $b$ ($\sqrt{\frac{3}{4}}$) : $b$ R ($\frac{1}{2}$) :: O A (1) : A T $(t) = \sqrt{\frac{1}{3}}$)

Whence

$$t^3 \left(= t \times t^2 = t \times \tfrac{1}{3}\right) = .1924500$$

$$t^5 \left(= t^3 \times t^2 = \frac{t^3}{3}\right) = .0641500$$

$$t^7 \left(= t^5 \times t^2 = \frac{t^5}{3}\right) = .0213833$$

$$t^9 \left(= t^7 \times t^2 = \frac{t^7}{3}\right) = .0071277$$

$$t^{11} \left(= t^9 \times t^2 = \frac{t^9}{3}\right) = .0023759$$

$$t^{13} \left(= t^{11} \times t^2 = \frac{t^{11}}{3}\right) = .0007919$$

$$t^{15} \left(= t^{13} \times t^2 = \frac{t^{13}}{3}\right) = .0002639$$

&c.

And therefore $\text{A R} = .5773502 - \frac{.1924500}{3} + \frac{.0641500}{5} - \frac{.0213833}{7} + \frac{.0071277}{9} - \frac{.0023759}{11} +$

$$+\frac{.0007919}{13}-\frac{.0002639}{15}+\frac{.0000879}{17}-\frac{.0000293}{19}$$

$$+\frac{.0000097}{21}-\frac{.0000032}{23}=.5235987:$$ which multiplied by 6 gives 3.141592 + for the length of the semi-periphery of the circle whose radius is unity.

At Article 126, certain forms of fluxions were pointed out, whose fluents are explicable by means of hyperbolical spaces, or a *table of logarithms:* which forms, it is observable, agree in every thing, but the signs (and constant quantities) with those exhibited above, for the arch of a circle. And these last, like them, may serve as so many (other) theorems for finding fluents by means of a *table of sines, tangents and secants.* But, as such a table is usually calculated to a radius of 1,000000, &c. (or unity) the following equations, derived from those above, being adapted to that radius, will be rather more commodius.

$$\text{Thus the fluent of}\left\{\begin{array}{c}\dfrac{\dot{w}}{\sqrt{2aw-w^2}}\\[2ex]\dfrac{\dot{w}}{\sqrt{a^2-w^2}}\\[2ex]\dfrac{a\dot{w}}{a^2+w^2}\\[2ex]\dfrac{a\dot{w}}{w\sqrt{w^2-a^2}}\end{array}\right\}\text{is equal to the arch whose}\left\{\begin{array}{l}\text{Versed-sine}\\[2ex]\text{Right-sine}\\[2ex]\text{Tangent}\\[2ex]\text{Secant}\end{array}\right\}\text{is }\frac{w}{a}\text{, and radius unity.}$$

The way of deducing these expressions, from the foregoing ones, is extremely easy: for, if *A* be put to denote the arch whose radius is unity, and whose versed-sine, right-sine, tangent, or secant is $\frac{w}{a}$ (according to the different cases here specified). Then, because similar arcs, of unequal circles, are as their radii, it will be $1 : a :: A : (aA)$ the length of the arch A R (*see the figure*). Therefore, the fluent of $\frac{a\dot{x}}{\sqrt{2ax-x^2}}$ (or $\frac{a\dot{w}}{\sqrt{2aw-w^2}}$, putting $w=x$) being $=aA$ (A R), that of $\frac{\dot{w}}{\sqrt{2aw-w^2}}$ must necessarily be $=A$: and in the very same manner the other forms are made out.

## EXAMPLE VII.

143. *Let the proposed Curve be the common Cycloid.*

Then, if the radius A O of the generating semi-circle* be denoted by $a$, we shall have B R $=\sqrt{2ax-x^2}$; and the fluxion thereof $=\frac{a\dot{x}}{\sqrt{2ax-x^2}}$: which being added to $\left(\frac{a\dot{x}}{\sqrt{2ax-x^2}}\right)$ the fluxion of A R or its equal R S (given by the preceding article) we thence get $\frac{2a\dot{x}-x\dot{x}}{\sqrt{2ax-x^2}}=\frac{\dot{x}\times\overline{2a-x}}{x^{\frac{1}{2}}\times\overline{2a-x}\rceil^{\frac{1}{2}}}=\frac{\dot{x}}{x^{\frac{1}{2}}}\times\overline{2a-x}\rceil^{\frac{1}{2}}$, for the true fluxion of the ordinate B S of the cycloid.

Hence $\dot{z}\ (\sqrt{\dot{x}^2+\dot{y}^2}\,^*)=\sqrt{\dot{x}^2+\frac{\dot{x}^2\times\overline{2a-x}}{x}}=\dot{x}\sqrt{\frac{2a}{x}}=\overline{2a}\rceil^{\frac{1}{2}}\times x^{-\frac{1}{2}}\dot{x}$; and consequently, by taking the fluent, $z=\overline{2a}\rceil^{\frac{1}{2}}\times\frac{x^{\frac{1}{2}}}{\frac{1}{2}}=2\sqrt{2ax}=$ the arch A S of the cycloid.

* See fig. Art. 132.

* Art. 135.

## EXAMPLE VIII.

144. *Wherein it is required to determine the Length of the Arch of the common Hyperbola.*

In this case (the semi-transverse axis being represented by $b$, and the semi-conjugate by $c$) we have $\frac{b^2y^2}{c^2} = 2bx + x^2$; and therefore $x = \frac{b\sqrt{c^2+y^2}}{c} - b$: hence $\dot{x} = \frac{by\dot{y}}{c\sqrt{c^2+y^2}}$, and $\dot{z}$ $(= \sqrt{\dot{y}^2 + \dot{x}^2})$ $\sqrt{\dot{y}^2 + \frac{b^2y^2\dot{y}^2}{c^2 \times \overline{c^2+y^2}}} = \dot{y}\sqrt{1 + \frac{b^2y^2}{c^4 + c^2y^2}}$; which, by converting $\frac{b^2y^2}{c^4+c^2y^2}$ into an *infinite series*, becomes $\dot{y}\sqrt{1 + \frac{b^2y^2}{c^4} - \frac{b^2y^4}{c^6} + \frac{b^2y^6}{c^8} - \frac{b^2y^8}{c^{10}}}$ &c. But still we have the square root to extract; in order thereto, let it be assumed $= 1 + Ay^2 + By^4 + Cy^6 + Dy^8$ &c. Then, by squaring and transposing (*Vide* Art. 98) there arises

$$\left.\begin{array}{l} 1 + 2Ay^2 + 2By^4 + 2Cy^6 + 2Dy^8 \text{ \&c.} \\ \qquad\qquad + A^2y^4 + 2ABy^6 + 2ACy^8 \text{ \&c.} \\ \qquad\qquad\qquad\qquad\qquad + B^2y^8 \text{ \&c.} \\ -1 - \frac{b^2}{c^4} \times y^2 + \frac{b^2}{c^6} \times y^4 - \frac{b^2}{c^8} \times y^6 + \frac{b^2}{c^{10}} \times y^8 \text{ \&c.} \end{array}\right\} = 0$$

Hence $A = \frac{b^2}{2c^4}$; $B = -\frac{b^2}{2c^6} - \frac{1}{2}A^2 = -\frac{b^2}{2c^6} - \frac{b^4}{8c^8}$; $C = \frac{b^2}{2c^8} - AB = \frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}$, &c. &c. Therefore $\dot{z}$ $(= \dot{y}\sqrt{1 + \frac{b^2y^2}{c^4}}$ &c. $= \dot{y} \times \overline{1 + Ay^2 + By^4 \text{ \&c.}} = \dot{y} + \frac{b^2}{2c^4} \times y^2\dot{y} - \left(\frac{b^2}{2c^6} + \frac{b^4}{8c^8}\right) \times$

$y^4\dot{y} + \left(\frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}\right) \times y^6\dot{y}$ &c. And consequently $z = y + \frac{b^2y^3}{6c^4} - \left(\frac{b^2}{c^2} + \frac{b^4}{4c^4}\right) \times \frac{y^5}{10c^4} + \left(\frac{b^2}{c^2} + \frac{b^4}{2c^4} + \frac{b^6}{8c^6}\right) \times \frac{y^7}{14c^6}$ &c.

By the very same way of proceeding, the arch of an ellipsis may be found, the equation of the two curves differing in nothing but their signs.

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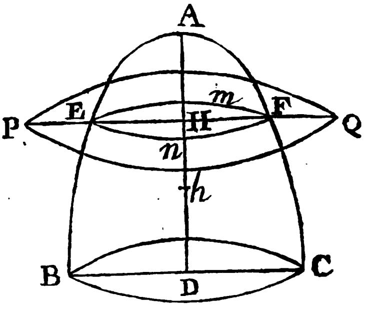
## SECTION IX.

### *The Application of Fluxions in investigating the Contents of Solids.*

145. LET A B C represent any solid; conceived to be generated (or described) by a plane P Q passing over it, with a parallel motion: let H $h$ (perpendicular to P Q) be taken to express the fluxion of A H $(x)$ or the velocity with which the generating plane is carried; also let the area of the part E $m$ F $n$ of the plane intercepted by, or contained in, the solid, be denoted by $A$: then it follows, from Art. 2 and 5, that the fluxion of the solid A E F, will be expressed by $A\dot{x}$. From whence, by expounding $A$ in terms of $x$ (according to the nature of the figure) and then taking the fluent, the content



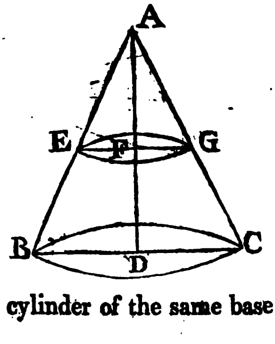
of the solid (which we shall always hereafter represent by $s$) will be given.

But, when the proposed solid is that arising from the revolution of any given curve A E B about A H D, as an axis, the fluxion ($\dot{s}$) of the solidity may be exhibited in a manner more convenient for practice: for,
* Art. 124. putting the area (3,141592 &c.*) of the circle, whose radius is unity, $= p$, and the ordinate E H $= y$, it will be $1^2 : y^2 :: p : (py^2)$ the area of the circle E$m$F$n$, which being wrote above instead of $A$, we have $\dot{s} = py^2\dot{x}$. The use of which will be sufficiently shown in the following Examples.

## EXAMPLE I.

### 146. *Let it be proposed to find the Content of a Cone* A B C.

Put the given altitude (A D) of the Cone $= a$, and the semi-diameter (B D) of its base $= b$: then, the distance (A F) of the circle E G, from the vertex A, being denoted by $x$, &c. we have, by similar triangles, as $a : b :: x :$ E F $(y) = \frac{bx}{a}$. Whence, in this case, $\dot{s}$ $(= py^2\dot{x}) = \frac{pb^2x^2\dot{x}}{a^2}$; and consequently $s = \frac{pb^2x^3}{3a^2}$; which, when $x = a$ (= A D) gives $\frac{pb^2a}{3}$ $(= p \times \text{BD}^2 \times \frac{1}{3}\text{AD})$ for the content of the whole cone A B C. Which appears from hence to be just $\frac{1}{3}$ of a cylinder of the same base and altitude.



## EXAMPLE II.

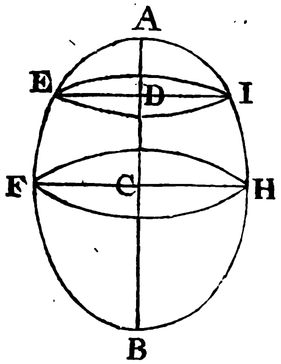
147. *Where, let the Solid proposed be a parabolic Conoid, or that arising from the Revolution of any Kind of Parabola about its Axis.*

Then, from the equation $a^{m-n}x^n = y^m$, of the generating curve, we get $y = a^{\frac{m-n}{m}} \times x^{\frac{n}{m}}$, and $\dot{s}$ $(= py^2\dot{x})$ $= pa^{\frac{2m-2n}{m}} \times \dot{x}x^{\frac{2n}{m}}$; and therefore $s = pa^{\frac{2m-2n}{m}} \times \frac{x^{\frac{2n}{m}+1}}{\frac{2n}{m}+1} = pa^{\frac{2m-2n}{m}} \times \frac{mx^{\frac{2n}{m}+1}}{2n+m} = pa^{\frac{2m-2n}{m}} \times x^{\frac{2n}{m}} \times \frac{mx}{2n+m} = py^2 \times \frac{mx}{2n+m}$ = the content of the solid; which therefore is to $(py^2x)$ the content of the circumscribing cylinder, as $m$ to $2n+m$. Whence the solid generated by the conical parabola (where $m=2$, and $n=1$) appears to be just $\frac{1}{2}$ of its circumscribing cylinder.

## EXAMPLE III.

148. *Let the proposed Solid* A F B H *be a Spheroid.*

In which case, putting the axis A B, about which the solid is generated, $=a$, and the other axis F H, of the generating ellipsis, $=b$, it follows, from the property of the ellipsis, that $a^2 : b^2 :: x \times \overline{a-x}$ (AD × BD) $: y^2$ (DE²) $= \frac{b^2}{a^2} \times \overline{ax - x^2}$: whence we have $\dot{s}$ $(= py^2\dot{x}$ *$) = \frac{pb^2}{a^2} \times \overline{ax\dot{x} - x^2\dot{x}}$; and * Art. 145.
$s = \frac{pb^2}{a^2} \times \overline{\tfrac{1}{2}ax^2 - \tfrac{1}{3}x^3}$ = the segment A I E. Which



when A D ($x$) = A B ($a$) becomes $\left(\frac{pb^2}{a^2} \times \overline{\tfrac{1}{2}a^3 - \tfrac{1}{3}a^3}\right)$ $\frac{1}{6}pab^2$ = the content of the whole spheroid. Where, if $b$ (F H) be taken = $a$ (A B) we shall also get $\frac{1}{6}pa^3$ for the true content of the sphere, whose diameter is $a$. Hence a sphere, or a spheroid, is $\frac{2}{3}$ of its circumscribing cylinder; for the area of the circle F H being expressed by $\frac{pb^2}{4}$, the content of the cylinder whose diameter is F H, and altitude A B, will therefore be $\frac{pb^2a}{4}$; of which $\frac{1}{6}pab^2$, is evidently two third parts.

## EXAMPLE IV.

149. *Let the Solid, whose Content you would find, be the hyperbolical Conoid.*

Then, from the equation, $y^2 = \frac{b^2}{a^2} \times \overline{ax + x^2}$, of the generating hyperbola, we have $\dot{s}$ $(py^2\dot{x}) = \frac{pb^2}{a^2} \times \overline{ax\dot{x} + x^2\dot{x}}$, and consequently $s = \frac{pb^2}{a^2} \times \overline{\tfrac{1}{2}ax^2 + \tfrac{1}{3}x^3}$ = the content of the conoid; which, therefore, is to $\left(\frac{pb^2}{a^2} \times \overline{ax + x^2} \times x\right)$ that of a cylinder of the same base and altitude, as $\frac{1}{2}a + \frac{1}{3}x$ to $a + x$. This ratio, if $x$ be extremely small, will become as 1 to 2 very nearly: whence it may be inferred, that the content of a very small part of any solid, generated by a curve, whose ray of curvature at the vertex is a finite quantity, is half that of a cylinder of the same base and altitude, very nearly: because any such curve, for a small distance, will differ insensibly from an hyperbola, whose radius of curvature, at the vertex, is the same.

This might have been inferred, either from the common parabolic conoid, or the spheroid, in the preceding Examples; but other observations would not allow room for it there.

## EXAMPLE V.

### 150. *In which the proposed Solid is that arising from the Rotation of the Cissoid of* Diocles, *about its Axis.*

Here, $y^2$ being $= \frac{x^3}{a-x}$,* we have $\dot{s}\ (py^2\dot{x}) = \frac{px^3\dot{x}}{a-x}$. But, in cases like this (where the denominator is rational and the variable quantity in the numerator of several dimensions, it will be necessary to divide the latter by the former, in order to obtain the fluent, by lessening the number of dimensions: thus, dividing $px^3\dot{x}$ by $-x+a$, according to the manner of compound quantities, the work will stand thus:

$$\begin{array}{lllll}
-x+a\,) & px^3\dot{x}-0 & & & (\,-px^2\dot{x}-pax\dot{x}-pa^2\dot{x} \\
 & \underline{px^3\dot{x}-pax^2\dot{x}} & & & \\
 & & +pax^2\dot{x}-0 & & \\
 & & \underline{+pax^2\dot{x}-pa^2x\dot{x}} & & \\
 & & & +pa^2x\dot{x}-0 & \\
 & & & \underline{+pa^2x\dot{x}-pa^3\dot{x}} & \\
 & & & \qquad\quad +pa^3\dot{x} &
\end{array}$$

Where the quotient being $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}$, and the remainder $pa^3\dot{x}$, the value of the given fraction $\frac{px^3\dot{x}}{a-x}$,

[*] Art. 56.

will therefore be truly expressed by $-px^2\dot{x} - pax\dot{x} - pa^2\dot{x} + \frac{pa^3\dot{x}}{a-x}$: whose fluent, properly corrected, is $-\frac{1}{3}px^3 - \frac{1}{2}pax^2 - pa^2x + pa^3 \times \text{hyp. log.}\ \frac{a}{a-x}$ *Vide* Art. 126.

## EXAMPLE VI.

151. *Let the Solid be that arising from the Rotation of the Conchoid of* Nicomedes *about its Axis.*

The sub-tangent $\frac{y\dot{x}}{\dot{y}}$ of this curve being $= \frac{-ab^2-y^3}{y\sqrt{b^2-y^2}}$ (*Vide* Art. 48 and 57) we have $\dot{x} = \frac{-ab^2\dot{y}-y^3\dot{y}}{y^2\sqrt{b^2-y^2}}$, and

* Art. 145. therefore $\dot{s}\ (py^2\dot{x}^*) = \frac{-pab^2\dot{y}-py^3\dot{y}}{\sqrt{b^2-y^2}} = -\frac{pab^2\dot{y}}{\sqrt{b^2-y^2}} - \frac{py^3\dot{y}}{\sqrt{b^3-y^2}}$. But, in order for the more easy finding the fluent thereof, put $\sqrt{b^2-y^2} = u$; and then, $y$ being $= \sqrt{b^2-u^2}$, and $\dot{y} = \frac{-u\dot{u}}{\sqrt{b^2-u^2}}$, we shall, by substitution, get $\dot{s} = \frac{pab^2\dot{u}}{\sqrt{b^2-u^2}} + p \times \overline{b^2\dot{u}-u^2\dot{u}}$.

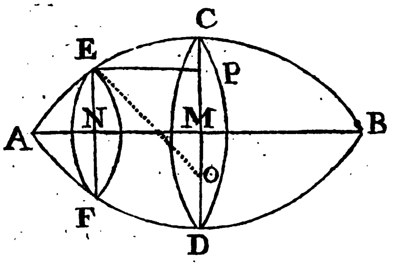
Whence, the fluent of $\frac{\dot{u}}{\sqrt{b^2-u^2}}$ being expressed by the arch ($A$) of the circle whose radius is unity and

† Art. 142. sine $\frac{u}{b}$,† the fluent of the whole expression will be $pab^2 \times A + p \times \overline{b^2u-\frac{1}{3}u^3}$. Which, when $y=0$, or $u=b$, gives ($pab^2 \times \frac{1}{2}p + p \times \frac{2}{3}b^3$) $pb^2 \times (\frac{1}{2}pa + \frac{2}{3}b)$ for the content of the whole solid, when its axis becomes infinite.

## EXAMPLE VII.

152. *Where it is required to find the Content of a parabolic Spindle; generated by the rotation of a given Parabola* A C B *about its Ordinate* A B.

Put C M (the abscissa of the given parabola) $= a$, and the semi-ordinate A M (or B M) $= b$; and, supposing E N F to be any section of the solid parallel to D C, let its distance M N (or E P) from D C, be denoted by $w$: then, by the property of the curve, we shall



have A M$^2$ $(b^2)$ : E P$^2$ $(w^2)$ :: C M $(a)$ : C P $= \frac{aw^2}{b^2}$: therefore E N $(=$ C M $-$ C P$) = a - \frac{aw^2}{b^2} = \frac{a \times \overline{b^2 - w^2}}{b^2}$, and consequently $p \times$ E N$^2$ $= \frac{pa^2}{b^4} \times \overline{b^4 - 2b^2w^2 + w^4}$ = the area of the section E F: which multiplied by $(\dot{w})$ the fluxion of M N, gives $\frac{pa^2}{b^4} \times \overline{b^4\dot{w} - 2b^2w^2\dot{w} + w^4\dot{w}}$ for the fluxion of the solidity,* whose fluent, $\frac{pa^2}{b^4} \times \overline{b^4w - \frac{2}{3}b^2w^3 + \frac{1}{5}w^5}$, * Art. 145.

when $w$ becomes $= b$, is $\left(\frac{8pa^2b}{15}\right)$ half the content of the solid.

## EXAMPLE VIII.

153. *Let the Solid* A C B D (*see the last figure*) *be a Spindle, generated by the Rotation of the Segment of a Circle,* A C B, *about its Chord, or Ordinate,* A B.

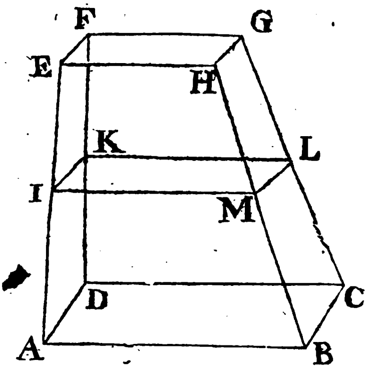
Then, if the radius O E be put $= r$, O M $= d$, and E P $= w$ &c. (as before) we shall have O P ($= \sqrt{\mathrm{OE}^2 - \mathrm{EP}^2}$) $= \sqrt{r^2 - w^2}$, and E N (O P $-$ O M) $= \sqrt{r^3 - w^2} - d$: therefore $\dot{s}$, in this case, is $= p\dot{w} \times \overline{\sqrt{r^2 - w^2} - d}\,^2 = p\dot{w} \times \overline{r^2 - w^2 + d^2 - 2d\sqrt{r^2 - w^2}}$ $= p\dot{w} \times \overline{r^2 - d^2 - w^2} - p\dot{w} \times \overline{2d\sqrt{r^2 - w^2} - 2d^2}$: whence, the fluent of the *part*, $p\dot{w} \times \overline{2d\sqrt{r^2 - w^2} - 2d^2}$ ($= 2dp \times \dot{w} \times \overline{\sqrt{r^2 - w^2} - d} = 2dp \times \dot{w} \times \mathrm{EN}$) being expressed by $2dp \times$ *area MNEC** the fluent of the *whole*, or the true value of $s$, will be expressed by $pw \times \overline{r^2 - d^2 - \tfrac{1}{3}w^2} - 2dp \times$ *area MNEC*, or by its equal $p \times \mathrm{MN} \times \overline{\mathrm{AM}^2 - \tfrac{1}{3}\mathrm{MN}^2} - 2p \times \mathrm{OM} \times$ *area MNEC*: which, when M N $=$ M A, gives $p \times \tfrac{2}{3}\mathrm{AM}^3 - 2p \times \mathrm{OM} \times$ *area ACM*, for the content of half the solid: where the *area ACM* may be found by Art. 124, or more easily by the common table of the areas of the segments of a circle; to be met with in most books of *gauging*.

* Art. 112

## EXAMPLE IX.

154. *Let it be proposed to find the Content of the Solid* A E G B; *whose four sides* AH, AF, CH, CF, *are plane Surfaces, and its Ends* A D C B, E F G H *given Rectangles, parallel to each other.*

Let the sides A B and A D, of the base, be denoted by $a$ and $b$; and those of the top (E H and E F) by $c$ and $d$ respectively; moreover, let $h$ express the perpen dicular height of the solid; and let $x$ (considered as variable) be the distance of (I L) any section thereof (parallel to the base) from the plane E G.



It is evident, from the nature of the figure, that the section I L is a rectangle; and that
$h : x :: AB-EH : IM-EH :: BC-HG : ML-HG$.

From these proportions we have I M − E H $= \frac{\overline{a-c}\times x}{h}$ and M L − H G $= \frac{\overline{b-d}\times x}{h}$: hence I M $= \frac{\overline{a-c}\times x}{h} + c$, and M L $= \frac{\overline{b-d}\times x}{h} + d$; and consequently the area of the rectangle ( I L ) $= \frac{\overline{a-c}\times\overline{b-d}}{h^2}\times x^2 + \frac{ad-2cd+cb}{h}\times x + cd$: which being multiplied by $\dot{x}$, and the fluent taken, there results $\frac{\overline{a-c}\times\overline{b-d}\times x^3}{3h^2} + \frac{\overline{ad-2cd+cb}\times x^2}{2h} + cdx$ for the content of I F G L.

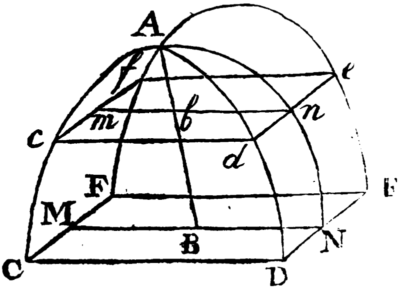
which, when $x = h$, becomes $\left(\frac{\overline{a-c}\times\overline{b-d}\times h}{3} + \frac{\overline{ad-2cd+cb}\times h}{2} + cdh = \overline{2ab+ad+bc+2cd}\times\frac{1}{6}h=\right)$ $\overline{AB\times AD+EH\times EF+\overline{AB+EH}\times\overline{AD+EF}}\times\frac{1}{6}h=$ the quantity proposed to be found.

If E F $(d)$ be supposed to vanish, and the lines E H and FG to coincide, the planes A E H B and D F G C will form an angle or ridge, at the top of the solid (resembling the roofs of some buildings, whose ends as well as sides run up sloping) and, in this case, the content, found above, will become more simple, being then expressed by $\overline{2ab+bc}\times\frac{1}{6}h$, or its equal $\overline{2AB+EH}\times AD\times\frac{1}{6}h$.

But, if EF be supposed $=$ EH, and AD $=$ AB, the solid will then be the frustrum of a square pyramid; and its content $= \overline{a^2+ac+c^2}\times\frac{1}{3}h, = \overline{AB^2+AB\times EH+EH^2}\times\frac{1}{3}h$: from whence, by taking $EH=0$, the content of the whole pyramid whose base is $AB^2$, and its altitude $h$, will also be given, being $=AB^2\times\frac{1}{3}h$.

## EXAMPLE X.

155. *Let the proposed Solid be that, commonly known by the Name of a Groin;* whose sections parallel to the base are, all squares, and whereof the two sections perpendicular to the Base, through the Middle of the opposite Sides, are Semi-circles.



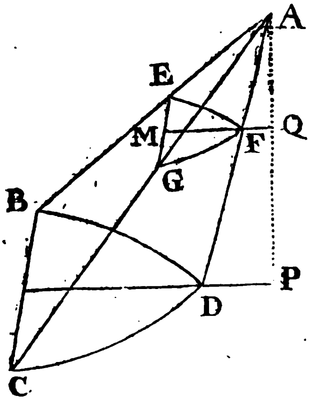
Let $bcdef$ be any section parallel to the base; and let its distance A $b$ from the vertex of the solid, be denoted by $x$; also let $a$ represent the radius A B (or B N) of the circular section A B N A, perpendicular to the base. Then, $bn$ being (by the property of the circle) $= \sqrt{2ax - x^2}$, the side of the square $df$, will be $= 2\sqrt{2ax - x^2}$, and therefore the area $= 4 \times \overline{2ax - x^2}$; whence $\dot{s} = 4\dot{x} \times \overline{2ax - x^2}$, and consequently $s = 4ax^2 - \frac{4x^3}{3}$: which, when $x = a$, becomes $\frac{\overline{2a}|^3}{3} =$ the content of the whole solid.

If the solid be a groin of any other kind, or such, that its two sections perpendicular to the base, through the middle of the opposite sides, are any other curves than semi-circles, the content may, still, be found in the same manner; and will be always in proportion to the solid generated by the revolution of the said curve about its axis, as a square is to its inscribed circle. But, if the foresaid perpendicular sections be curves of different kinds, the sections parallel to the base will no longer be squares, but rectangles; whose sides are the corresponding (double) ordinates of the respective curves. Thus, for instance, let one section be a circle and the other a parabola, whose ordinates, to the common abscissa $x$, are expressed by $\sqrt{dx - x^2}$ and $\sqrt{ax}$, respectively; then the sides of the rectangular section, parallel to the base of the groin, will be $2\sqrt{dx - x^2}$ and $2\sqrt{ax}$: whence the area of that section is $= 4x\sqrt{ad - ax}$, and therefore $\dot{s} = 4x\dot{x}\sqrt{ad - ax}$: where, by taking the fluent,[*] $s =$ $\frac{16d^2\sqrt{ad} - a^{\frac{1}{2}} \times \overline{d - x}|^{\frac{3}{2}} \times \overline{16d + 24x}}{15} =$ the true content of such a solid.

[*] Art. 83.

## EXAMPLE XI.

156. *Where the Solid* B A C D *proposed is a kind of Cone, or Pyramid;* formed by conceiving Right-lines to be drawn from every Point in the Perimeter of any given Plane B D C, to a given Point, or Vertex, A above that Plane.



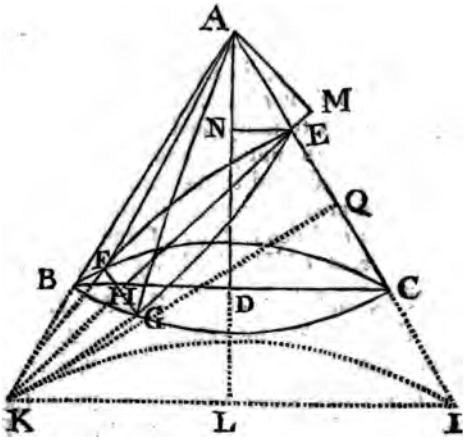
Let E F G be any section parallel to B D C, whose perpendicular distance (A Q) from the vertex let be denoted by $x$; moreover, let the whole given altitude (AP) of the solid be put $= a$, and the area of the base B D C (which is also supposed given) $= b$.

In the first place, it is easy to conceive that the planes B D C and E F G must be similar: and therefore, since similar figures are to each other as the squares of their like sides, or dimensions, it follows that AP² $(a^2)$ : AQ² $(x^2)$ : : BDC $(b)$ : E F G $= \frac{bx^2}{a^2}$.

Whence $\dot{s} = \frac{bx^2\dot{x}}{a^2}$, and consequently $s = \frac{bx^3}{3a^2} = \frac{ba}{3}$, when $x = a$. Therefore the solidity of a cone or pyramid, let the figure of its base be what it will, is always had by multiplying the area of the base by $\frac{1}{3}$ of the altitude.

## EXAMPLE XII.

157. *Where it is proposed to find the Content of the Ungula* E F G C, cut off from a given Cone, A B C, by a Plane E F G passing through the Base thereof



Let A D be the perpendicular height of the cone, also let A M be perpendicular to H E, the axis of the section F E G, and let F A G be another section of the cone, through F G and the vertex A.

Since the solids CAFG and E A F G, whose bases are FCG, and FEG, come under the form specified in the preceding example, their contents will therefore be expressed by F C G × $\frac{1}{3}$A D and FEG × $\frac{1}{3}$AM respectively: whose difference, $\frac{FCG \times AD - FEG \times AM}{3}$, is the solidity of the *ungula* C E F G: where the bases FCG and FEG being conic sections, their areas will be given by Art. 115, 124, and 129, from whence the whole will be known. Thus, if H E be supposed parallel to A B, the section F E G, then being a parabola, its area will be $= \frac{2}{3} \times$ F G × E H :* whence the solidity of the

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* Art. 115.

segment E F G A is $=\frac{2}{9}\times FG\times EH\times AM$: which being deducted from that of C F G A (found by help of the common table of circular segments). the remainder will be the content of the *ungula*. But, if the axis E H produced, cuts A B, the section F E G will be a segment of an ellipsis E F K G; whose conjugate axis (supposing E N and K L perpendicular to A D) is $= 2\sqrt{EN\times KL}$.* Now, in order to compute the content, the easiest way, in this case, let the ratio of E H to E K (which is given by trigonometry) be expressed by that of *m* to unity, and let the ratio of C H to C B, be as *n* to unity: and from the common table of *segments* (adapted to the circle whose diameter is unity) let the areas answering to the versed sines *m* and *n*, be taken and denoted by *M* and *N* respectively: then the area of F E G being $= M\times EK\times 2\sqrt{EN\times KL}$, and that of F C G $=N\times BC^2$,† the content of the *ungula*, by substituting these values, will become $=\frac{1}{3}N\times BC^2\times AD-\frac{1}{3}M\times EK\times AM\times 2\sqrt{EN\times KL}$: but, since $AM : AE :: KQ$ (perpendicular to AC) : KE; and $AN : AE :: KQ : KI$, it follows, by equality, that $AM\times KE = AN\times KI$; whence the content of the *ungula* is also expressed by $\frac{1}{3}N\times BC^2\times AD-\frac{1}{3}M\times AN\times KI\times 2\sqrt{EN\times KL}$. Which, if H be supposed to coincide with B, and K I with B C, will become $\left(\frac{0.78539\ \&c.}{3}\times BC^2\times AD-\frac{0.78539\ \&c.}{3}\times AN\times BC\times 2\sqrt{EN\times BD}\right)=0.26179\ \&c.\times BC\times\overline{BC\times AD-2AN\times\sqrt{EN\times BD}}$.

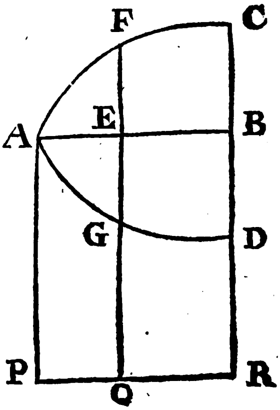
* Art. 41.

† Art. 124 & 130.

When the section E F G is an hyperbola, its area may be found by means of a table of logarithms (instead of a table of segments) whence the content of the *ungula* will likewise be had in that case.

## EXAMPLE XIII.

158. *Let* A F C, *or* A G D, *be a Curve of any kind: whose Area, and the Content of the Solid arising from its Rotation about its Axis or Ordinate* A B, *are both known; it is proposed to find, from thence, the Content of the Solid generated by the Revolution of that Curve about any other Line* P R *parallel to the said Axis or Ordinate* A B.



Let A P, F Q, and C R be all perpendicular to A B and to the axis of motion P Q R; also let A P (or E Q) $= a$, A E, considered as variable, $= w$, the area A F E, or A E G $= M$, and the solid arising from its revolution about A B, $= N$. It is plain that the area of the circle generated by Q F will be $= p \times FQ^2$ * $= p \times \overline{a + EF}|^2 = pa^2 + 2pa \times EF + p \times EF^2$; from which deducting the area $pa^2$, generated by Q E, the remainder, $2pa \times EF + p \times EF^2$ will be the area of the annulus generated by E F: whence the fluxion of the solid generated by A E F is truly represented by $2pa \times EF \times \dot{w} + p\dot{w} \times EF^2$ :† and, in the same manner, it will appear that the fluxion of the solid generated by A E G is $2pa \times EG \times \dot{w} - p\dot{w} \times EG^2$. But the fluent of $EF \times \dot{w}$ (or $EG \times \dot{w}$) is $=$ the area *(M)* of A E F (or A E G),‡ and that of $p\dot{w} \times EF^2$ (or $p\dot{w} \times EG^2$) equal to *(N)* the given solid arising from that area; § therefore the fluent of the *whole*, or the solidity required, is $2paM + N$, in the former case, and $2paM - N$ in the latter; where $2pa$,

* Art. 145.

† Art. 145.

‡ Art. 112.

§ Art. 145.

in either case, expresses the periphery of the cylinder described by A B, about the axis of rotation P R.

Hence, if A B C and A B D are equal and similar to each other, then the value of $M$, &c. being the ſame in both cases, it follows that the content of the solid generated by A F G will be expressed by $2pa \times 2M$, or $2pa \times$ area A F G.

Now, if (for example's sake) A C D be supposed a circle, whose semi-diameter is $d$, the area of that circle being $= pd^2$, the solid generated by its revolution (representing the ring of an anchor) will therefore be $= 2pa \times pd^2 = 2p^2ad^2$. But if you would know the content of the part generated by the upper semicircle B A C, or the lower one B A D, let the content

* Art. 148. $\left(\frac{4pd^3}{3}\right)$* of a sphere whose semi-diameter is $d$, be wrote for $N$, in each of the two foregoing expressions, and you will then get $p^2ad^2 + \frac{4pd^3}{3}$ and $p^2ad^2 - \frac{4pd^3}{3}$.

Again, if A F C and A G D be taken as right-lines, you will have $M = \frac{AB \times BC}{2}$, (or $\frac{AB \times BD}{2}$) and

† Art. 146. $N = p \times BC^2 \times \frac{1}{3} AB$ (or $p \times BD^2 \times \frac{1}{3} AB$) †: hence the solid generated by the triangle A B C is $(= 2pa \times \frac{AB \times BC}{2} + \frac{p}{3} \times BC^2 \times AB) = p \times AB \times BC \times \overline{RB + \frac{1}{3} BC}$; and that generated by A B D $(= 2pa \times \frac{AB \times BD}{2} - \frac{p}{3} \times BD^2 \times AB) = p \times AB \times BD \times \overline{RB - \frac{1}{3} BD}$.

Lastly, let A B C (or A B D) be considered as a parabola, whose ordinate is A B, and axis C B (or D B):

‡ Art. 115. then M being here $= \frac{2}{3} AB \times BC$ (or $\frac{2}{3} AB \times BD$)‡

§ Art. 152. and $N = \frac{8p}{15} \times AB \times BC^2$§ (or $\frac{8p}{15} \times AB \times BD^2$)

it follows that the solid generated by ABC will be ($= 2pa \times \frac{2}{3} AB \times BC + \frac{8p}{15} \times AB \times BC^2) = 4p \times AB \times BC \times \frac{5BR + 2BC}{15}$, and that generated by $ABD = 4p \times AB \times BD \times \frac{5BR - 2BD}{15}$.

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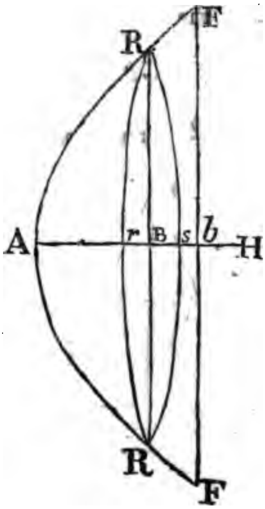
## SECTION X.

### *The Use of Fluxions in finding the Superficies of Solid Bodies.*



159. LET FAF represent a solid generated by the revolution of any given curve AF about its axis AH; also let a circle, whose diameter is the variable line (or ordinate) RBR, be conceived to move uniformly from A towards FF, and to dilate itself so, on all sides at the same time, as to generate, by its periphery, the proposed superficies RAR: then, the length of that periphery, or the generating line, being expressed by 3,141592 * &c. × RR ($= 2py$) and the celerity with which it moves by $\dot{s}$ † the fluxion of the superficies RAR, or the space that

* Art. 142.

† Art. 135.

*would be* uniformly generated in the time of describing $\dot{z}$, will therefore be truly represented by $2py\dot{z}$.

Hence, if $w$ be taken to represent the whole surface R A R, generated from the beginning (according to the method observed in the three last Sections) we shall

* Art. 135. have $\dot{w} = 2py\dot{z} = 2py\sqrt{\dot{x}^2 + \dot{y}^2}$;* whence $w$ itself may be found.

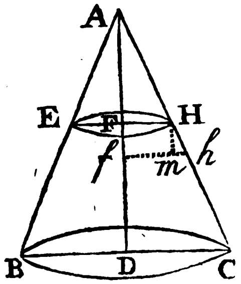
## EXAMPLE I.

160. *Let it be proposed to determine the convex Superficies of a Cone* A B C.

Then, the semi-diameter of the base (B D or C D) being put $= b$, the slanting line, or hypothenuse, A C $= c$, and F H (parallel to D C) $= y$, &c. we shall, from the similarity of the triangles A D C and H $m$ $h$,

† Art. 159. have $b : c :: \dot{y}\ (mh) : \dot{z}\ (\mathrm{H}h) = \frac{c\dot{y}}{b}$ : whence $\dot{w}\ (2py\dot{z})$†

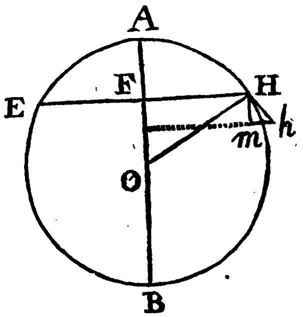
$= \frac{2pcy\dot{y}}{b}$; and consequently $w = \frac{pcy^2}{b}$. This, when $y = b$, becomes $= pcb = p \times \mathrm{DC} \times \mathrm{AC} =$ the convex superficies of the whole cone A B C: which therefore is equal to a rectangle under half the circumference of the base and the slanting line.



## EXAMPLE II.

### 161. *Let the Solid, whose Surface you would find, be a Sphere* A E B H.

In which case, putting the radius O H $=a$, A F $=x$, H $m=\dot{x}$, &c. we shall (by reason of the similar triangles O H F and H $m$ $h$*) have $y$ (F H) : $a$ (OH) : : $\dot{x}$ (H $m$) : $\dot{z}$ (H $h$) $= \frac{a\dot{x}}{y}$ : therefore $\dot{w}$ $(2py\dot{z}) = 2pa\dot{x}$; and consequently the superficies ($w$) itself $=2pax=$ A F × *Periph.* AEBH. Which, if the whole sphere be taken, will become A B × *Periph.* AEBH $=$ four times the area B E A H O.



Hence the superficies of a sphere is equal to four times the area of its greatest circle: and the convex superficies of any segment thereof, is to that of the *whole*, as the axis (or thickness) of the segment to the diameter of the sphere.

## EXAMPLE III.

### 162. *Wherein let the parabolic Conoid be proposed.*

The equation of the generating parabola being $ax=y^2$, or $x=\frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore

$$\dot{z} \left( = \sqrt{\dot{y}^2 + \dot{x}^2}^{*} \right) = \sqrt{\dot{y}^2 + \frac{4y^2\dot{y}^2}{a^2}} = \frac{\dot{y}\sqrt{a^2 + 4y^2}}{a}\dagger$$

hence $\dot{w}$ $(2py\dot{z}) = \frac{2py\dot{y}}{a} \times \overline{a^2 + 4y^2}|^{\frac{1}{2}}$; whereof the fluent is $\frac{p\times\overline{a^2+4y^2}|^{\frac{3}{2}}}{6a}$; which corrected (by supposing

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* Art. 68.

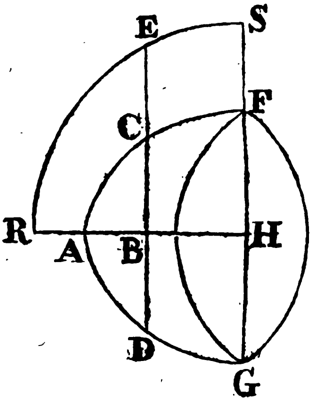
† Art. 135.

* Art. 79. $y=0$ *) gives $\frac{p\times\overline{a^2+4y^2}|^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$, for the superficies sought.

## EXAMPLE IV.

163. *Let it be required to determine the Superficies of a Spheroid.*

Let A C F H G represent one half of the proposed spheroid, generated by the rotation of the semi-ellipsis F A G, about its axis A H; put AH $=a$, FH (or HG) $=c$, B H $=x$, B C $=y$, F C $=z$, and the superficies generated by F C (or G D) $= w$: then, from the nature of the ellipsis, we have $y = \frac{c}{a}\sqrt{a^2-x^2}$; whence



Art. 135 $\dot y = -\frac{cx\dot x}{a\sqrt{a^2-x^2}}$, and consequently $\dot z$ ($= \sqrt{\dot x^2+\dot y^2}$ *)

$= \sqrt{\dot{x}^2 + \frac{c^2x^2\dot{x}^2}{a^2 \times \overline{a^2 - x^2}}} = \frac{\dot{x}\sqrt{a^4 - \overline{a^2 - c^2} \times x^2}}{a\sqrt{a^2 - x^2}} = \frac{\dot{x}\sqrt{a^4 - b^2x^2}}{a\sqrt{a^2 - x^2}}$ = (by putting (the eccentricity) $\sqrt{a^2 - c^2} = b$) $= \frac{b\dot{x}\sqrt{\frac{a^4}{b^2} - x^2}}{a\sqrt{a^2 - x^2}}$ : therefore, in this case, $\dot{w}$ $(2py\dot{z})$ $= \frac{2pbc\dot{x}}{a^2}\sqrt{\frac{a^4}{b^2} - x^2}$; whose fluent, in an infinite series, is $2pcx \times \overline{1 - \frac{b^2x^2}{2.3a^4} - \frac{b^4x^4}{2.4.5a^8} - \frac{3b^6x^6}{2.4.6.7a^{12}}}$. But the same fluent may be, *otherwise*, very easily exhibited by means of the area of a circle: for, if from the center H, with a radius equal to $\frac{a^2}{b}$, a circle S E R be described, and the ordinate B C be produced to intersect it in E, it is evident that B E $= \sqrt{\frac{a^4}{b^2} - x^2}$, and that the fluxion of the area E S H B will be expressed by $\dot{x}\sqrt{\frac{a^4}{b^2} - x^2}$; which being to $\frac{2pbc\dot{x}}{a^2} \times \sqrt{\frac{a^4}{b^2} - x^2}$, the fluxion before found, in the constant ratio of 1 to $\frac{2pbc}{a^2}$, their fluents must therefore be in the same ratio; and so the latter, expressing the superficies C F G D, will consequently be $= \frac{2pbc}{a^2} \times$ B E S F H $= 2p \times \frac{\text{F H}}{\text{H S}}$ × B E S F H.

This solution, it may be observed, obtains only in case of an *oblong* spheroid, generated by the rotation of the ellipsis about its greater axis; for, in an *oblate* spheroid, generated about the lesser axis, the value of $b$ $(\sqrt{a^2-c^2})$ will be impossible; since, in this case, HF is greater than HA. But if we *here* put $b = \sqrt{c^2 - a^2}$; and $d = \frac{a^2}{b}$, the value of $\dot{w}$ (found above) will become $= \frac{2pbc\dot{x}}{a^2}\sqrt{\frac{a^4}{b^2} + x^2} = \frac{2pc\dot{x}}{d}\sqrt{d^2 + x^2} = \frac{2pc}{d} \times \dot{x}\sqrt{d^2 + x^2}$: whose fluent may be brought out by help of a table of logarithms: for, let the variable part $\dot{x}\sqrt{d^2 + x^2}$ be transformed to $\left(\frac{\dot{x} \times \overline{d^2+x^2}}{\sqrt{d^2 + x^2}} = \frac{d^2\dot{x}+x^2\dot{x}}{\sqrt{d^2+x^2}} = \frac{d^2x\dot{x}+x^3\dot{x}}{\sqrt{d^2x^2+x^4}} =\right)$ $\frac{\frac{1}{2}d^2x\dot{x}+x^3\dot{x}}{\sqrt{d^2x^2+x^4}} + \frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, so that the numerator of the first term $\frac{\frac{1}{2}d^2x\dot{x}+x^3\dot{x}}{\sqrt{d^2x^2+x^4}}$ (now in a given ratio to the fluxion of the quantity under the radical sign) may be had by the common rule;* by which means we get $\frac{1}{2}\sqrt{d^2x^2 + x^4}$, for the true fluent of the said term; to which adding the fluent of the other term $\frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, or $\frac{\frac{1}{2}d^2\dot{x}}{\sqrt{d^2 + x^2}}$ (given by Art. 126), there arises $\frac{1}{2}x\sqrt{d^2 + x^2} + \frac{1}{2}d^2 \times$ hyp. log. $(x + \sqrt{d^2 + x^2})$, for the fluent of $\dot{x}\sqrt{d^2 + x^2}$: and this, corrected † and multiplied by $\frac{2pc}{d}$ gives $\frac{pcx}{d} \times \sqrt{d^2 + x^2} + pcd \times$ hyp. log. $\frac{x + \sqrt{d^2 + x^2}}{d}$, for the superficies in this case, where the proposed spheroid is an oblate one.

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* Art. 77.

† Art. 78.

## EXAMPLE V.

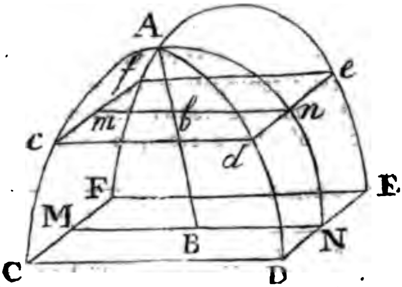
164. *Let the Solid, whose Superficies is sought, be the hyperbolical Conoid.*

Let the semi-transverse axis of the generating hyperbola $= a$, the semi-conjugate $= c$, and the distance of any ordinate from the center thereof $= x$; then from the nature of the curve you will have $y = \frac{c}{a}\sqrt{x^2 - a^2}$; whence $\dot{y} = \frac{cx\dot{x}}{a\sqrt{x^2 - a^2}}$, $\dot{z} = \frac{\dot{x}\sqrt{\overline{a^2 + c^2} \times x^2 - a^4}}{a\sqrt{x^2 - a^2}}$, and $\dot{w}$ $(2py\dot{z}) = \frac{2pc\dot{x}}{a^2} \times \sqrt{\overline{a^2 + c^2} \times x^2 - a^4}$; which last value, if $d^2$ be put $= \frac{a^4}{a^2 + c^2}$, will be more commodiously expressed by $\frac{2pc\dot{x}}{d}\sqrt{x^2 - d^2}$: whereof the fluent, by proceeding as in the latter part of the foregoing example, will come out $= \frac{pcx\sqrt{x^2 - d^2}}{d} - pcd \times$ *hyp. log.* $(x + \sqrt{x^2 - d^2})$: which corrected (by taking $x = a$) becomes $\frac{pcx}{d}\sqrt{x^2 - d^2} - pc^2 - pcd \times$ *hyp. log.* $\frac{x + \sqrt{x^2 - d^2}}{a + \frac{cd}{a}}$, the true measure of the required superficies.

## EXAMPLE VI.

165. *Let it be proposed to find the Superficies of the Solid called a Groin.* (*Vide* Art. 155).

Let *bcdef* be any section of the solid parallel to the base thereof, and let $x$ denote its distance from the vertex A, also put $z$ equal to the corresponding arch A $n$ of the semi-circular section N $n$ A, &c. whose radius A B or B N let be denoted by $a$.



It appears from Art. 161, that $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - x^2}}$: which value, multiplied by $(2\sqrt{2ax - x^2})$ that of $de$ $(=2bn)$ gives $2a\dot{x}$* for the fluxion of one of the four equal convex superficies by which the solid is bounded. Hence the whole superficies (excluding the base) comes out $= 8a^2$: which therefore is exactly equal to twice the base.

If the solid be supposed a groin of any other kind, such that its two equal sections, through the middle of the opposite sides, are other curves than circles, the superficies may *still* be had in the same manner; and will be always in proportion to the superficies arising from the revolution of either of the said equal curves about its axis, as a square is to its inscribed circle. Thus, the superficies of a parabolic conoid being =

$$\frac{p \times \overline{a^2 + 4y^2}\,\big|^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$$ (by Art. 162) the convex

superficies of the groin, supposing the generating curve A $n$ N to be a parabola, will therefore be =

$$\frac{4 \times \overline{a^2 + 4y^2}\,\big|^{\frac{3}{2}}}{6a} - \frac{4a^2}{6}.$$

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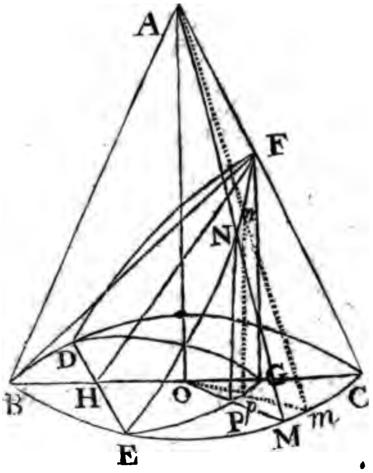
* Art. 159.

## EXAMPLE VII.

166. *Wherein let it be required to find the convex Superficies of a conical* Ungula E C F D; *formed by a Plane* D F E, *passing through the Base of the Cone.*

Let a right-angled triangle A O M (whose base O M is the radius of the circle B D C E) be supposed to revolve about the axis A O; whilst a right-line N P, drawn perpendicular to O M from the intersection of A M and the arch E F D, traces out, upon the base of the cone, the curve line E P G D.



If M P O A N and *m p* O A *n* be considered as two positions of the generating triangle indefinitely near to each other, it is evident that the space M A *m*, generated by A M, will be to the space M O *m*, generated by O M, as A M to O M, or O B. Whence, M N and M P being proportional parts of A M and O M (because N P is parallel to A O) it is likewise plain that the spaces MN*nm* and MP*pm*, generated by those parts, will be to each other in the same ratio of A M to O B. And, since this every where holds, it follows that the whole space (ENM) &c. generated by MN, will be to that (EPM) generated by PM, as AM to OB: and so the whole required superficies (generated by AM) is truly represented by $\frac{AM}{OB}$ × area EPGDCE.

But now, to find this area E P G D C E, it is observable that the area of the plane D F E (being the segment of a conic-section) is given, by Art. 115, 129 or 130. And it is very easy to apprehend and demonstrate that the area so given will be to that of E G D H, as the radius to the co-sine of the angle of the inclination of the said plane to the base, or as H F to HG. Therefore, seeing EGDH is $= \frac{HG}{HF} \times EFD$, we have E P G D C E ($= ECDHE - EGDH$) $= ECDHE - \frac{HG}{HF} \times EFD$; and consequently $\frac{AM}{OB} \times$ EPGCDE $= \frac{AM}{OB} \times ECDHE - \frac{AM \times HG}{OB \times HF} \times EFD =$ the convex superficies that was to be found.

If the point H be supposed to coincide with B, ECDHE will become the *whole* circle CB; and EDF will become a whole ellipsis, whose greater axis is BF, and its lesser axis $= 2\sqrt{OB \times OG}$.* Therefore, the area of the former figure will be expressed by $p \times BO^2$,† and that of the latter by $p \times \frac{1}{2}BF \times \sqrt{OB \times OG}$; and so the convex superficies of the part BFC will be $\left(= \frac{AM}{OB} \times p \times BO^2 - \frac{AM \times BG}{OB \times BF} \times p \times \frac{1}{2}BF \times \sqrt{OB \times OG}\right) = p \times AM \times OB - p \times AM \times \frac{1}{2}BG \times \sqrt{\frac{OG}{OB}}$: which being deducted from ($p \times AM \times OB$) the superficies of the whole cone B A C, there rests $p \times AM \times \frac{1}{2}BG \times \sqrt{\frac{OG}{OB}}$, for the superficies of the oblique cone BAF; which from hence is also given.

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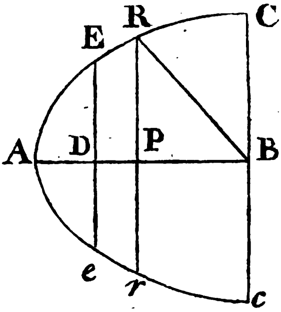
* Art. 41.

† Art. 124.

## SCHOLIUM.

167. In most of the examples, delivered in the four last sections, the part of the proposed *figure* next the vertex, whether a curve, solid, or superficies, is first found; from whence, by taking the altitude ($x$) of that part equal to ($a$) the altitude given, the content of the *whole* is deduced: but, if the content of the lower segment (BCED) of any figure (ABC) arising by taking away a part (ADE) next the vertex, be required; then the difference between the *whole* and the part taken away (found as before explained) will be the quantity sought.



Thus, for example, let ABC be the common *parabola*, and let it be proposed to find the content of the part BCED included between any two ordinates BC ($b$) and DE ($c$) at a given distance BD ($d$) from each other: then, the equation of the curve being $ax=y^2$, we have $\dot{x}=\frac{2y\dot{y}}{a}$, and therefore $y\dot{x}=\frac{2y^2\dot{y}}{a}$,* Art. 112.

whose fluent $\frac{2y^3}{3a}$ is a general expression for the area comprehended between the vertex and the ordinate $y$: whence, expounding $y$ by $b$ and $c$ successively, we get $\frac{2b^3}{3a}$ and $\frac{2c^3}{3a}$ for the corresponding values of ABC and ADE; whose difference $\frac{2b^3-2c^3}{3a}$ is the required area BCED: but, to express the same independent of $a$, it will be, by the property of the curve, $b^2 : c^2 :: AB : AD$;

whence, by division, $b^2 : b^2 - c^2 :: \mathrm{AB} : \mathrm{BD}\ (d)$ and consequently $\frac{b^2 - c^2}{d} = \frac{b^2}{\mathrm{AB}} = a$; which first value being wrote instead of $a$, there results $\mathrm{BCED} = \frac{\overline{2b^3 - 2c^3} \times d}{3b^2 - 3c^2} = \frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$.

After the same manner, the segments of other figures may be found; but in many cases they will be more readily had from a direct investigation, without either finding the whole, or the part taken away.

Thus, in the case above, if the excess of any ordinate R P above D E $(c)$ be denoted by $w$, we shall have, by the property of the curve, $b^2 - c^2$ ($\mathrm{BC}^2 - \mathrm{DE}^2$) : $\overline{c + w}|^2 - c^2$ ($\mathrm{RP}^2 - \mathrm{DE}^2$) :: D B $(d)$ : $\mathrm{DP} = \frac{d \times \overline{2cw + w^2}}{b^2 - c^2}$; whose fluxion $\left(d \times \frac{2c\dot{w} + 2w\dot{w}}{b^2 - c^2}\right)$ multiplied by $c + w$ ($=$ P R) gives $d \times \frac{2c^2\dot{w} + 4cw\dot{w} + 2w^2\dot{w}}{b^2 - c^2}$, for the fluxion of the area DPRE: whereof the fluent (which is $2dw \times \left.\frac{c^2 + cw + \frac{1}{3}w^2}{b^2 - c^2}\right)$ will, when $w = b - c$ (or $\mathrm{RP} = \mathrm{BC}$) be truly expounded by $\frac{2d \times \overline{b - c} \times \overline{\frac{1}{3}b^2 + \frac{1}{3}bc + \frac{1}{3}c^2}}{b^2 - c^2}$ or its equal $\frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$; *the same as before.*

*Again*, for another example, let C E D $e$ $c$ be considered as the lower frustrum of an hemisphere, whose center is the point B: then, B P being *here* denoted by $w$, we shall have $y^2$ $(= \mathrm{BR}^2 - \mathrm{BP}^2) = b^2 - w^2$, and consequently $py^2\dot{w}$ * $= p \times \overline{b^2\dot{w} - w^2\dot{w}}$; whose fluent ($p \times \overline{b^2w - \frac{1}{3}w^3} = \frac{1}{3}pw \times \overline{3b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + y^2} = \frac{1}{3}p \times BP \times \overline{2BC^2 + PK^2}$) is the true content of the part C E D $e\,c$; which will also hold when the figure is a spheroid.

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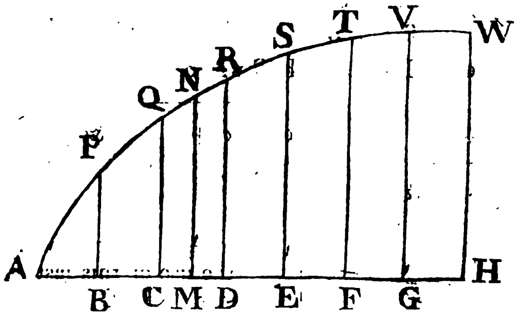
* Art. 145.

This last method of finding the content of a portion of a figure remote from the vertex, will be of service, when the general value, for the *whole*, cannot be expressed without an infinite series; because such a series, in that case, not converging, becomes useless.*

By dividing the whole proposed figure, A H W, into a number of such portions, H V, G T, F S, &c. the content thereof may be obtained, when to find it at once, by a series, commencing from the vertex, would be altogether impracticable.



But, to render such an operation as short and easy as may be, it will be proper to find each part (DQ, &c.) of the figure, by means of a series proceeding both ways, from the middle ordinate (M N) between the two corresponding extremes (C Q and D R)

Thus, let the value of MN (found by the property of the curve) be denoted by $a$; and let the value of D R, in a series, be represented by $a + bx + cx^2 + dx^3 + ex^4$ $fx^5 +$ &c. where $x = $ M D; then the area M D R N will be represented by the fluent of $\dot{x} \times \overline{a + bx + cx^2 + dx^3 +}$

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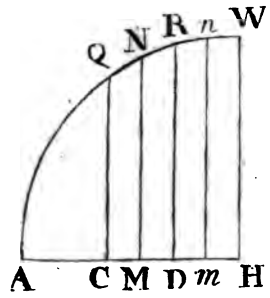
* Art. 93.

&c. or by $x \times \overline{a + \frac{bx}{2} + \frac{cx^2}{3} + \frac{dx^3}{4} + \&c.}$ And by writing $-x$ instead of $x$, the ordinate C Q will be expressed by $a - bx + cx^2 - dx^3$ &c. and the area M C Q N by $x \times \overline{a - \frac{bx}{2} + \frac{cx^2}{3} - \frac{dx^3}{4} - \frac{ex^4}{5}}$ &c. whence the area C D R Q is $= 2x \times \overline{a + \frac{cx^2}{3} + \frac{ex^4}{5} + \frac{gx^6}{7} + \&c.}$ Therefore, if D E, E F, F G, and G H be supposed, each = B C ($2x$) and the areas D S, E T, &c. (found as above) be denoted by $2x \times \overline{\overset{\prime}{a} + \frac{\overset{\prime}{c}x^2}{3} + \frac{\overset{\prime}{e}x^4}{5}}$ &c. and $2x \times \overline{\overset{\prime\prime}{a} + \frac{\overset{\prime\prime}{c}x^2}{3} + \frac{\overset{\prime\prime}{e}x^4}{5}}$ &c. respectively, it follows that the area C R + D S + E T will be represented by $2x \times \overline{a + \overset{\prime}{a} + \overset{\prime\prime}{a}}$ &c. $+ \frac{2}{3}x^3 \times \overline{c + \overset{\prime}{c} + \overset{\prime\prime}{c}}$ &c. $+ \frac{2}{5}x^5 \times \overline{e + \overset{\prime}{e} + \overset{\prime\prime}{e}}$ &c.



An example will show the use of this last expression: let C H W Q be a portion of a quadrant H A W of a circle, whose base H C (conceived to be divided into four equal parts) is equal to half the radius A H, represented by *unity*. Then, putting C M (= D M = D $m$ = $m$ H = $\frac{1}{8}$) $= x$, H M ($= \frac{3}{8}$) $= p$, and H $m$ ($= \frac{1}{8}$) $= q$, we have, by the property of the circle, $a$ (M N) $= \sqrt{\text{HN}^2 - \text{HM}^2} = \sqrt{1 - p^2}$, and

D R $(=\sqrt{HR^2 - HD^2}) = \sqrt{1 - \overline{p - x}|^2} = \sqrt{1 - p^2 + 2px - x^2} = \sqrt{a^2 + 2px - x^2}$; which, in a series, is $(= a + \frac{2px - x^2}{2a} - \frac{\overline{2px - x^2}|^2}{8a^3} + \&c.)$ $= a + \frac{px}{a} - \overline{\frac{1}{2a}} + \overline{\frac{p^2}{2a^3}} \times x^2$ &c. Therefore, in this case, $b = \frac{p}{a}$, $c = -\overline{\frac{1}{2a} + \frac{p^2}{2a^3}}$, &c. Which value of $c$, by writing $1 - a^2$ for its equal $p^2$, will be reduced to $-\frac{1}{2a^3}$. From whence it is also evident that $\acute{c} = -\frac{1}{2\acute{a}^3}$ (supposing $\acute{a}\ (mn) = \sqrt{1 - q^2}$)

Consequently $2x \times \overline{a + \acute{a} + \ddot{a}}$ &c. $+ \frac{2}{3}x^3 \times \overline{c + \acute{c} + \ddot{c}}$ &c. $+ \frac{2}{5}x^5 \times \overline{e + \acute{e} + \ddot{e}}$ &c. $(= \overline{a + \acute{a}} \times 2x + \overline{c + \acute{c}} \times \frac{2x^3}{3}) = \overline{\sqrt{\frac{55}{64}} + \sqrt{\frac{63}{64}}} \times \frac{1}{4} -$

$$\overline{\frac{1}{\frac{2 \times 55}{64}\sqrt{\frac{55}{64}}} + \frac{1}{\frac{2 \times 63}{64}\sqrt{\frac{63}{64}}}} \times \frac{2}{64 \times 8 \times 3} =$$

$$\frac{\sqrt{55} + \sqrt{63}}{32} - \frac{1}{3 \times 55\sqrt{55}} - \frac{1}{3 \times 63\sqrt{63}} =$$

$$\frac{\sqrt{55} + \sqrt{63}}{32} - \frac{\sqrt{55}}{3 \times 55 \times 55} - \frac{\sqrt{63}}{3 \times 63 \times 63} =$$

0,48730 = the area C H W Q, that was to be found.

This example, chosen as an illustration of the foregoing method, may indeed be wrought the common way; whence the very same conclusion is brought out

(*Vide* Art. 124). But that method is also applicable to any other case, whether the part proposed be near to the vertex, or remote from it; and whether the figure itself be a curve, solid or superficies; since the measure thereof may, always, be expressed by the area of a curve.

There is another way, well known to mathematicians, whereby the area of a curve may be determined, by means of a number of equidistant ordinates; which method, derived from *that of differences*, may, also, be used to good purpose, in cases like those above specified: but, it having been treated of by several *others*, and also in my "*Mathematical Dissertations*,"* the reader will excuse me, if no further notice is taken of it here.

## SECTION XI.

### *Of the Use of* FLUXIONS *in finding the Centers of Gravity, Percussion, and Oscillation of Bodies.*

168. THE Center of Gravity is that Point of a Body, by which, if it were suspended, it would rest *in Equilibrio*, in any Position.

LEMMA.

169. *Let p, q, r, s, &c. be any Number of given Weights, hanging at an inflexible Line (or Rod)* AM *suspended* in Equilibrio, *in an horizontal Position, at the Point* O; *to determine the Position of that Point.*

Since (*by Mechanics*) the force of any weight (*p*) to raise the opposite end (M) of the balance, is as that weight drawn into its distance (B O) from the fulcrum, we shall, from the equality of these forces, have $p\times OB+q\times OC+r\times OD=s\times OE+t\times OF$,

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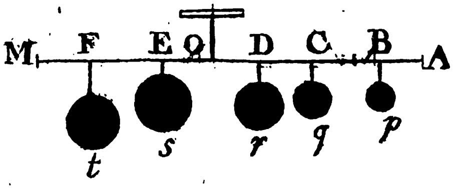
* A new edition of this work is in preparation.



that is $p\times\overline{AO-AB}+q\times\overline{AO-AC}+r\times\overline{AO-AD}=s\times\overline{AE-AO}+t\times\overline{AF-AO}$, and consequently $AO=\dfrac{p\times AB+q\times AC+r\times AD+s\times AE+t\times AF}{p+q+r+s+t}$.

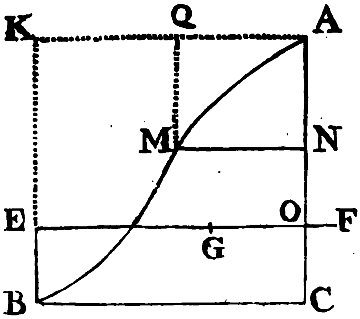
From which it appears, that, *if each weight be multiplied by its distance from the end (or any given point) of the axis, the sum of all the products divided by the sum of all the weights, will give the distance of the center of gravity from that end (or point).*

*Note.* The products here mentioned are, usually, called *the forces* of their respective weights; not in respect to their action at the center O (which is expressed by a different quantity) but with regard to the effects they have in the conclusion, or the value of A O; which appear to be in that ratio.

## PROPOSITION I.

170. *To determine the Center of Gravity of a Line, Plane, Superficies, or Solid* (admitting the three former capable of being affected by Gravity).

Let A M B C be the proposed figure, and G the center of gravity thereof; through which, parallel to the horizon, let the line E F be drawn, intersecting A C, at right-angles, in O; also let A K and N M be perpendicular to A C, and parallel to E F.



171. Case 1. *If the figure* A M B C *be a plane;* let it be supposed to rest *in Equilibrio* upon the line E F; and then, if the line M N be considered as a weight, its force (*defined above*) will be expressed by M N drawn into its distance (A N) from the end of the axis A C; that is by $yx$ (supposing, as usual, A N $= x$ and M N $= y$). This, therefore, multiplied by the fluxion of A N, gives $yx\dot{x}$ for the fluxion of the force of the plane A M N; whose fluent, when $x =$ A C, expresses the force of the whole plane, or the sum of all the products of the ordinates (or weights) by their respective distances from A K: which fluent being, therefore, divided by the area A B C, or the fluent of $y\dot{x}$ (according to the foregoing Lemma) the quotient $\left(\frac{Flu.\, yx\dot{x}}{Flu.\, y\dot{x}}\right)$ will give (A O) the distance of the center of gravity from the line A K.

172. Case 2. *If the figure be a solid;* let M N be a section thereof by a plane perpendicular to the horizon; then, the area of that section being denoted by $A$, the force thereof (considered as above) will be expressed by $Ax$, and the fluxion of the force of the solid A M N by $Ax\dot{x}$; whose fluent, divided by the content of the body, or the fluent of $A\dot{x}$, gives A O in this case. But, if the solid be the half (or the whole) of that arising from the rotation of a curve A M B about its axis A C; then (putting $p$ for the area of the circle
* Art. 145. whose radius is unity) $A$ will become $= \frac{1}{2}py^2$;* and consequently A O $= \frac{Flu.\, \frac{1}{2}py^2x\dot{x}}{Flu.\, \frac{1}{2}py^2\dot{x}} = \frac{Flu.\, y^2x\dot{x}}{Flu.\, y^2\dot{x}}$.

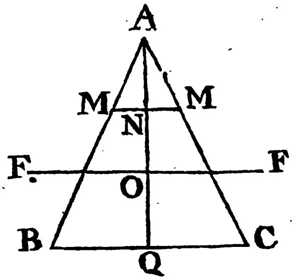
173. Case 3. *If the figure proposed be the curve-line* AMB; then, the force of a particle at M being expressed by A N or M Q $(x)$ we shall (putting $AM=z$) have $\frac{\text{flu. } x\dot{z}}{z} = AO$.

174. Case 4. *But if the Figure given be the Superficies generated by the Rotation of* A M B *about* A C.

Then, the periphery of the circle generated by the point M being $= 2py$, it follows that $\frac{\text{flu. } 2pyx\dot{z}}{\text{flu. } 2py\dot{z}} = \frac{\text{flu. } yx\dot{z}}{\text{flu. } y\dot{z}} = AO$.

## EXAMPLE I.

175. *Let the Figure proposed be the isosceles Triangle* ABC.



It is evident the center of gravity (O) will be somewhere in the perpendicular A Q: and, if $AQ=a$, $BC=b$, $AN=x$, and $MM=y$; then $y$ being $=\frac{bx}{a}$, we shall have, *by Case* 1, AO $\left(=\frac{\text{flu. } yx\dot{x}}{\text{flu. } y\dot{x}}\right) = \frac{\text{flu. } x^2\dot{x}}{\text{flu. } x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2x}{3} = \frac{2}{3}$ A Q, when $x = AQ$, and consequently $OQ = \frac{AQ}{3}$.

In the very same manner, the center of gravity of any other (plane) triangle will appear to be at $\frac{1}{3}$ of the altitude of the triangle.

## EXAMPLE II.

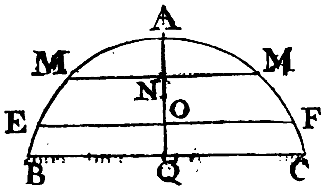
### 176. *Let the Figure proposed be a Parabola of any Kind; whereof the Equation is* $y = \frac{x^n}{a^{n-1}}$.

Here, $\frac{\text{flu. } yx\dot{x}}{\text{flu. } y\dot{x}} = \frac{\text{flu. } x^{n+1}\dot{x}}{\text{flu. } x^n\dot{x}} = \frac{n+1}{n+2} \times x =$ the distance of the center of gravity from the vertex of the curve.

## EXAMPLE III.

### 177. *Let* B A C *be a Segment of a Circle.*

Then, if the radius thereof be put $= r$, we shall have $y$ (N M) $= \sqrt{2rx - x^2}$: whence the fluent of $yx\dot{x}$ $(x\dot{x}\sqrt{2rx - x^2})$ will, by Art. 163, be found $= -\frac{\overline{2rx - x^2}|^{\frac{3}{2}}}{3} + r \times$ area A N M; which divided by A N M, gives $r - \frac{\text{NM}^3}{3 \times \text{area ANM}} = \text{AO}$.* This therefore, when B A C is a semi-circle, becomes $= \frac{576}{1000} \times r$, nearly.



But, with respect to the center of gravity of the arch B A C; we have, *flu.* $x\dot{z}$, (*by Case* 3) $=$ fluent of $\frac{rx\dot{x}}{\sqrt{2rx - x^2}} = r = \overline{\text{AM} - \text{MN}}$; and consequently A O *here* $= r - \frac{r \times \text{MN}}{\text{AM}}$.

* Art. 171.

## EXAMPLE IV.

178. *Let* A B C (see the preceding figure) *represent a Segment of a Sphere, or Spheroid.*

In which case, denoting the axis of the sphere, or spheroid, by $a$, and the other axis of the generating curve, when an ellipsis, by $b$, we have $y^2=\frac{b^2}{a^2}\times\overline{ax-x^2}$;

and therefore $\frac{flu.\ y^2x\dot{x}}{flu.\ y^2\dot{x}}$ * $=\frac{flu.\ \overline{ax-x^2}\times x\dot{x}}{flu.\ \overline{ax-x^2}\times\dot{x}}=$

$$\frac{\frac{1}{3}ax^3-\frac{1}{4}x^4}{\frac{1}{2}ax^2-\frac{1}{3}x^3}=\frac{\frac{1}{3}ax-\frac{1}{4}x^2}{\frac{1}{2}a-\frac{1}{3}x}=\frac{x\times\overline{4a-3x}}{6a-4x}=\text{A O.}$$

If the solid be an hyperbolical conoid, the distance (A O) of its center of gravity from the vertex, will also be exhibited by the expression here brought out, when the negative signs are changed to positive ones.

179. In those cases where the figure cannot be divided into two parts, equal and like to each other (as a curve is by its axis, &c.) the position of two lines E O, $eo$ *(see the ensuing figure)* must be determined, *as above*; in whose intersection (G) the center of gravity will be found.

## EXAMPLE V.

*Let* A B C *be a Semi-parabola of any kind; whereof the Equation is* $y=\frac{x^n}{a^{n-1}}$.

It appears, from *Ex.* 2, that (A O) the distance of E G O from the vertex, is expressed by $\frac{n+1}{n+2}\times$ A C: but to find the position of $o$ G $e$ (perpendicular to E O) let M $n$ be parallel to $eo$, or A C; then, A N being $=x$, and N M $(y)=\frac{x^n}{a^{n-1}}$, if A C be denoted by $b$, we

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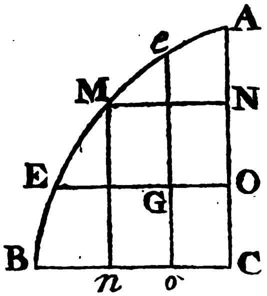
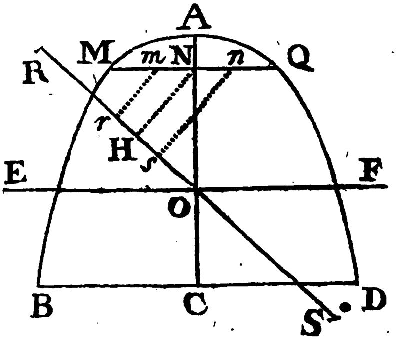
* Art. 172.



shall have M $n=b-x$, and M $n\times$ N M $\times\dot{y}=\overline{b-x}\times\frac{x^n}{a^{n-1}}\times\frac{nx^{n-1}\dot{x}}{a^{n-1}}=\frac{nbx^{2n-1}\dot{x}-nx^{2n}\dot{x}}{a^{2n-2}}$, for the fluxion of the sum of the forces in this case (*Vide* Art. 171), whose fluent $\left(\frac{nbx^{2n}}{2na^{2n-2}}-\frac{nx^{2n+1}}{\overline{2n+1}\times a^{2n-2}}\right.$ $=\frac{x^{2n}}{a^{2n-2}}\times\overline{\frac{b}{2}-\frac{nx}{2n+1}}=y^2\times\overline{\frac{b}{2}-\frac{nx}{2n+1}}=y^2\times\frac{b}{4n+2}$, or $\frac{\text{B C}^2\times\text{A C}}{4n+2}$ (when $x=b$) divided by the area A B C $\left(=\frac{\text{B C}\times\text{A C}}{n+1}\right)$ gives $\frac{n+1}{4n+2}\times$ B C for the true value of C $o$, or O G. Which, in case of the common parabola, where $n=\frac{1}{2}$, and where AO $\left(\frac{n+1}{n+2}\times\text{AC}\right)=\frac{3}{5}$ AC, will become $=\frac{3}{8}$ C B.

Before I leave this subject it may not be improper to take notice, *that*, whatever line you found your calculations upon, by supposing the figure to rest, *in*

*Equilibrio*, upon that line, the very same point, for the place of the center of gravity, will be determined.



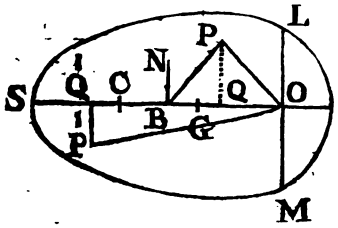
180. Thus, let O be the point in the axis A C, of a given curve B A D, determined, as above, by supposing the figure to rest upon E F perpendicular to A C; and let R S be any other line passing through the point O; then I say the sum of the *momenta* of the particles on each side of R S will, *also*, be equal. For, if from two points, in any ordinate M Q, equally distant from the middle point N, two perpendiculars *m r* and *n s* be let fall upon R S, the efficacy of those two points, in respect to RS, will be represented by $mr + ns$, or its equal 2NH (supposing NH also perpendicular to RS). Whence the efficacy of all the particles in M Q, will be expressed by their number multiplied by N H, or by M Q × N H: which is to their efficacy (M Q × O N) when referred to the line E F, in the constant ratio of N H to O N, or of the sine of the angle R O N to radius. Whence it is evident that the force of all the ordinates (or the whole curve) in the former case, must be to that in the latter, in the same ratio: but the said force, in the one case, is equal to nothing by hypothesis, therefore it must be likewise so in the other: and consequently the sum of the *momenta* of the particles, on each side of R S, equal to each other.

Much after the same manner the thing may be proved, in a solid: whence it will appear that there is actually such a (fixed) point in a body as the center of gravity is defined to be: which, however evident from mechanical considerations, is not so easy to demonstrate, *geometrically*, from the resolution of forces.

## PROPOSITION II.

### 181. *To determine the Center of Percussion of a Body.*

The Center of Percussion is that point, in the axis of suspension of a vibrating (or revolving) body, at which it may be stopt, by an immoveable obstacle, so as to rest thereon *in equilibrio* as it were, without acting upon the center of suspension.



Let O be the point of suspension, G the center of gravity, and S L M a section of the body, by the plane wherein the axis of suspension O G S performs its motion; to which section let all the particles of the body be conceived to be transferred in such parts thereof where they would be projected into (*orthographically*) by lines parallel to the axis of motion; which supposition will neither affect the place of the center of gravity nor the angular motion of the body.

Since the angular velocity of any particle P is as the distance, or radius, O P, its force in the direction, PB, perpendicular to OP, will be expressed by $P\times OP$. Therefore the efficacy of that force upon the axis, at B, in the perpendicular direction B N (supposing the axis stopt at C the center of percussion) will be $P\times OP\times\frac{OP}{OB}$, whose power to turn the body about the point C is therefore as $P\times OP\times\frac{OP}{OB}\times BC = P\times\frac{OP^2\times BC}{OB} = P\times\frac{OP^2\times\overline{OC-OB}}{OB} = P\times\frac{OP^2\times OC}{OB} - P\times OP^2$; which, if P Q be made perpendicular to

O S, will at last (because $\frac{OP^2}{OB} = OQ$) be reduced to $P \times OQ \times OC - P \times OP^2$. By the very same argument, the force of any other particle $\acute{P}$ will be denoted by $\acute{P} \times O\acute{Q} \times OC - \acute{P} \times O\acute{P}^2$ &c. &c. But, as all these forces must destroy one another (by the nature of the problem) the sum of all the quantities $P \times OQ \times OC$ $\acute{P} \times O\acute{Q} \times OC$, &c. must therefore be = the sum of all the quantities $P \times OP^2$, $\acute{P} \times O\acute{P}^2$ &c. and consequently $OC = \frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.\ \&c.}{P \times OQ + \acute{P} \times O\acute{Q} + \&c.\ \&c.}$. But (by the preceding proposition) the sum of all the quantities $P \times OQ + \acute{P} \times O\acute{Q} + \&c.$ is equal to $OG \times$ by the content of the body. Therefore O C is likewise =

$$\frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.\ \&c.}{OG \times body}.$$

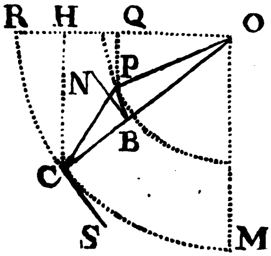
*The same otherwise.*

Since the force of the particle P, in the perpendicular direction N B, is defined by $P \times \frac{OP^2}{OB}$, or its equal, $P \times OQ$, the sum of all the quantities $P \times OQ$, $\acute{P} \times O\acute{Q}$, &c. &c. will express the force which, acting at C perpendicular to O S, is sufficient to stop the body, without the center of suspension O being any way affected: this sum, therefore, drawn into O C ($= OC \times P \times OQ + \acute{P} \times O\acute{Q} + \&c.\ \&c.$) is as the efficacy of the said force to turn the body about the point O. But the force of the particle P, in the direction B N being $P \times \frac{OP^2}{OB}$, its efficacy to turn the body about the same point (the contrary way) is as $P \times OP^2$; and consequently the efficacy of all the particles as the sum of all the quantities $P \times OP^2$, $\acute{P} \times O\acute{P}^2$ &c &c. Therefore (action and re-action being equal) we have $OC \times \overline{P \times OQ + \acute{P} \times O\acute{Q}} + \&c. = P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.$ *the same as before.*

For the Center of Oscillation, it will be requisite to premise the following

### Lemma.

182. *Suppose two exceeding small Weights* C *and* P, *acting on each other by means of an inflexible Line (or Wire)* P C *to vibrate in a vertical Plane* R O P C M, *about the Center* O; *it is required to determine how much the Motion of the one is affected by the other.*



Let CH and PQ be perpendicular to the horizontal line OR; also let PB and CS be perpendicular to OP and O C respectively.

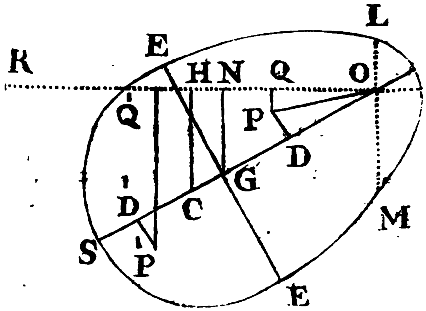
If the force of gravity be denoted by unity, the forces acting in the directions CS and PB, whereby the weights, in their descent, are accelerated, will, according to the resolution of forces, be represented by $\frac{OH}{OC}$ and $\frac{OQ}{OP}$. Moreover, since the velocities are always in the ratio of the radii O C and O P, if the foresaid forces were to be in that ratio, or that of P was to become $\frac{OH}{OC} \times \frac{OP}{OC}$, instead of $\frac{OQ}{OP}$; I say, in that case, it is plain, the weights would continue their motion without affecting each other, or acting at all on the line of communication P C (or P B). Whence, the excess of $\frac{OQ}{OP}$ above $\frac{OH}{OC} \times \frac{OP}{OC}$ must be the accelerative force whereby the weight P acts upon the line (or wire) P C, in the direction P B; which multiplied by the weight P gives $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}$ for the absolute force in that direction: which, therefore, in the perpendicular direction N B, is $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}} \times \frac{OP}{OB}$; whereof the part acting upon C, being to the whole as O B to O C, is truly defined by $P \times \overline{\frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}}$. *Q. E. I.*

If P be supposed to act upon C by means of PC (instead of P B) the conclusion will be no way different: for, let F (to shorten the operation) be put to denote the force $(P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}$ in the direction P B, found above, then the action thereof upon P C (according to the principles of mechanics) will be expressed by $F \times \frac{\textit{radius}}{\textit{co-s.}\ CPB}$; which therefore in the direction S C, perpendicular to OC, is $F \times \frac{\textit{radius}}{\textit{co-s.}\ CPB} \times \frac{\textit{S.}\ PCO}{\textit{radius}} = \frac{\textit{S.}\ PCO}{\textit{co-s.}\ CPB} = \frac{\textit{S.}\ PCO}{\textit{S.}\ OPC} = F \times \frac{OP}{OC}$, *the very same as before.*

## PROPOSITION III.

183. *To determine the Center of Oscillation of a Body.*

The center of oscillation is that point, in the axis (or line) of suspension of a vibrating body, into which if the whole body was contracted, the angular velocity and the time of vibration would remain unaltered.



Let L M S be a section of the body by a plane, perpendicular to the horizon and the axis of motion, passing through the center of gravity G and the point of suspension O; and suppose all the particles of the body to be transferred to this section, in such places of it, as they would be projected into *(orthographically)* by perpendiculars falling thereon. (Which supposition will no way affect the conclusion, the angular motion continuing the same). Moreover let C be the center of oscillation, or that point in the axis O S where a particle of matter (or a small weight) may be placed so as to be neither accelerated nor retarded by the action of the other particles of matter situate in the plane. Then, if, from C and any other point P in the plane L M S, two perpendiculars C H and P Q be let fall upon the horizontal line O R, the force of a particle (or weight) at P to accelerate the weight at C, will *(according to the foregoing Lemma)* be represented by P ×

$\overline{\frac{OQ}{OC} - \frac{OH\times OP^2}{OC^3}}$: which, supposing G N perpendicular to O R, will also be expressed by P × $\overline{\frac{OQ}{OC} - \frac{ON}{OG}\times\frac{OP^2}{OC^2}}$, or its equal P × $\frac{OQ\times OG\times OC - ON\times OP^2}{OG\times OC^2}$. In the very same manner the force of any other particle $\acute{P}$ will be represented by $\acute{P}\times\frac{O\acute{Q}\times OG\times OC - ON\times O\acute{P}^2}{OG\times OC^2}$ &c. &c.

Therefore the forces of all the particles destroying each other *(by hypothesis)* the sum of all the quantities $P\times\overline{OG\times OQ\times OC - ON\times OP^2}$ $+\acute{P}\times\overline{OG\times O\acute{Q}\times OC - ON\times O\acute{P}^2}$ &c. &c. must be equal to nothing: whence $P\times OG\times OQ\times OC + \acute{P}\times OG\times O\acute{Q}\times OC$ &c. &c. $= P\times ON\times OP^2 + \acute{P}\times ON\times O\acute{P}^2$ &c. &c. and consequently $OC = \frac{ON}{OG}\times\frac{P\times OP^2+\acute{P}\times O\acute{P}^2+\text{\&c.}}{P\times OQ+\acute{P}\times O\acute{Q}+\text{\&c.}}$ But (by Art. 171 and 172) the sum of all the quantities $P\times OQ+\acute{P}\times O\acute{Q}$ &c. is equal to the content of the body multiplied by the distance (O N) of the center of gravity G from the line L M (perpendicular to O C); whence O C is also $=\frac{ON}{OG}\times\frac{P\times OP^2+\acute{P}\times O\acute{P}^2\,\text{\&c. \&c.}}{ON\times \mathit{body}}=\frac{P\times OP^2+\acute{P}\times O\acute{P}^2\,\text{\&c. \&c.}}{OG\times \mathit{body}}$ Which expression continuing the same in all inclinations of the axis O S, the point C, thus determined is a fixed point, agreeable to the definition; and appears to be the same with the center of percussion; see Art. 181.

### COROLLARY.

184. If PD, $\acute{P}\acute{D}$ &c. be perpendicular to OS, the numerator of the fraction found above, will become $P\times(OG^2+GP^2-2OG\times GD)+\acute{P}\times(OG^2+G\acute{P}^2+2OG\;G\acute{D})$ + &c. &c. (since $OP^2=OG^2+GP^2-2OG\times GD$ &c.) Which, because all the quantities $P\times-2OG\times GD+\acute{P}\times 2OG\times G\acute{D}$ &c. or $P\times-GD+\acute{P}\times G\acute{D}$ &c. (by the nature of the center of gravity) destroy one another, will be *barely* $=P\times\overline{OG^2+GP^2}+\acute{P}\times\overline{OG^2+G\acute{P}^2}+\text{\&c. \&c.}=\overline{P+\acute{P}+\text{\&c.}}\times OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c.}=\textit{mass}\times OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c.}$ Whence it is evident that OC is, also,

$$\left(=\frac{\textit{mass}\times OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c. \&c.}}{\textit{mass}\times OG}\right)$$

$$=OG+\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c.}}{\textit{mass}\times OG};$$ and consequently

$$CG=\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\text{\&c. \&c.}}{\textit{mass}\times OG}$$

Whence it appears that, *if a body be turned about its center of gravity, in a direction perpendicular to the axis of the motion, the place of the center of oscillation will remain unaltered;* because the quantities $P\times GP^2$, $\acute{P}\times G\acute{P}^2$ are no way affected by such a motion of the body.

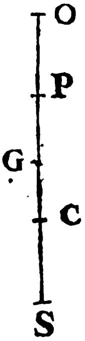
It also appears *that the distance of the center of gravity from that of oscillation* (if the plane of the body's motion remains unaltered) *will be reciprocally as the distance of the former from the point of suspension. Therefore, if that distance be found when the point of suspension is in the vertex, or so posited, that the operation may become the most simple, the value thereof in any other proposed position of that point will likewise be given, by one single proportion.*

185. But now, to show how these conclusions may be reduced to practice, we must first of all observe, that *the Product of any particle of the Body by the Square of its distance from the axis of motion is* (here) *called the force thereof* (its efficacy to turn the body about the said axis being in that ratio). According to which, and the first general value of O C, it appears that, *if the sum of all the forces be divided by the product of the body into the distance of the center of gravity from the point of suspension, the quotient thence arising will give the distance of the center of percussion, or oscillation from the said point of suspension.*

The manner of computing the divisor has been already explained; it remains therefore to show how the sum of all the forces in the numerator may be collected: which will admit of several cases. Wherein, to avoid a multiplicity of words, I shall always express the distance of the center of gravity from the point of suspension by $g$, and the distance of the center of percussion, or oscillation, from the same point, by $C$.

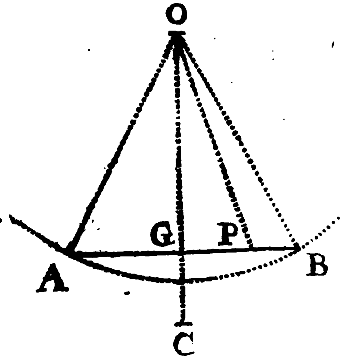
### 186. Case 1. *Let* O S *be a Line suspended at one of its Extremes.*

Then, if the part O P (considered as variable) be denoted by $x$, the force of $\dot{x}$ particles, at P, will *(as above)* be defined by $\dot{x} \times x^2$: whose fluent ($\frac{1}{3} x^3$) therefore expresses the force of all the particles in O P (or the sum of all the products, under each particle, and the square of its distance from O the point of suspension). This quantity, therefore (when $x$ be-



comes = O S) being divided by O S × ½ O S (according to the foregoing rule or observation) we get $\left(\frac{\frac{1}{3}OS^3}{\frac{1}{2}OS^2}=\right)$ ⅔ O S for the value of *C*, the true distance of the center of oscillation (or percussion) from the point of suspension.

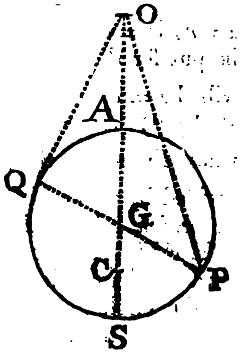
187. Case 2. *Let* A B *be a Line, vibrating in a vertical Plane, having its two Extremes* A *and* B *equally distant from the Point of Suspension* O.



If O G (perpendicular to A B) be put $=a$, and $GP=x$, the force of $\dot{x}$ particles at P, will be denoted by $\dot{x}\times\overline{a^2+x^2}=\dot{x}\times$ O P²:* whose fluent divided by $ax$ (or P G × O G) gives $\left(\frac{a^2x+\frac{1}{3}x^3}{ax}\right)a+\frac{x^2}{3a}=OG+\frac{BG^2}{3OG}=C$, when $x$ becomes = G B.

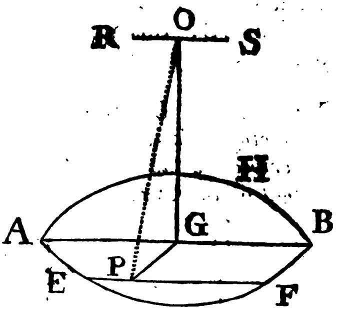
188. Case 3. *Let* APSQ *be a circle, vibrating in a vertical Plane.* Let P Q be any diameter thereof; then O P² + O Q² being = 2 O G² + 2 P G², the sum of the forces of two particles at P and Q (putting $OG=a$, and $AG=r$) will be $=\overline{a^2+r^2}\times 2$; whence it is evident that the sum of the forces of all the particles, in the whole periphery, will be expressed by *their number* $\times\,\overline{a^2+r^2}$, or by $\overline{a^2+r^2}\times$ *periph.* A P S Q: which,

* Art. 185.



if $p$ be put = 3.141 &c. will be = $\overline{a^2 + r^2} \times 2pr = 2pa^2r + 2pr^3$. Hence the force of the circle itself is also given, being = fluent of $\overline{2pa^2r + 2pr^3} \times \dot{r} = pa^2r^2 + \frac{1}{2}pr^4 = \overline{a^2 + \frac{1}{2}r^2}$ × circle APSQ. Now, if the two expressions thus found be divided by $a$ × *periph.* APSQ, and $a$ × *circle* APSQ respectively,* we shall have $a + \frac{r^2}{a}$ and $a + \frac{r^2}{2a}$, for the two corresponding values of $C$.

### 189. Case 4. *Let* A H B E *be a Circle having its Plane (always) perpendicular to the Axis of Suspension* OG.



Let AGB be that diameter of the circle which is parallel to the axis of motion RS; and let EF be any chord parallel to AB and RS; whose distance G P from the center of the circle, let be denoted by $x$; putting O G = $a$, and A G = $r$: then, by the nature of the circle, E F = $2\sqrt{r^2 - x^2}$; whose distance O P, from the axis of motion R S, is also given = $\sqrt{a^2 + x^2}$. Hence it appears that the force of all the particles in the line E F (*defined in Art.* 185) will be represented by $\overline{a^2 + x^2} \times 2\sqrt{r^2 - x^2}$. Therefore $\dot{x} \times \overline{a^2 + x^2} \times 2\sqrt{r^2 - x^2}$ is the fluxion of the force of the plane ABFE; whose fluent (when

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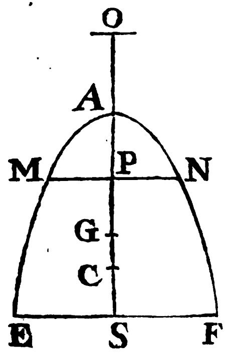
* Art. 185.

$x=r$) is $=\overline{a^2+\frac{1}{4}r^2}\times$ *area* $AEFBG$; which, if $p$ be put for the area of the circle whose radius is unity, will be $=\overline{a^2+\frac{1}{4}r^2}\times\frac{1}{2}pr^2$; whereof the double ($pa^2r^2+\frac{1}{4}pr^4$) is the force of the whole circle AEFH: whose fluxion $2par\dot{r}+pr^3\dot{r}$ (supposing $r$ variable) being divided by $\dot{r}$, we likewise get $2pa^2r+pr^3$ ($=\overline{a^2+\frac{1}{2}r^2}\times$ *periph.* $AEFH$) for the force of the periphery AEFH. But the center of gravity, whether we regard the circle itself or its periphery, is in the center of the circle; therefore the distance of the center of oscillation from the point of suspension, will in these two cases be exhibited by $a+\frac{r^2}{4a}$ and $a+\frac{r^2}{2a}$ respectively.

If the circle, instead of being perpendicular to GO, either coincides, or makes a given angle with it, the value of $C$ will come out exactly the same; provided the diameter AB still continues parallel to the axis of motion RS: this appears from Art. 184, and may be, otherwise, very easily demonstrated.

190. Case 5. *Let the Figure proposed be a Curve* AEF, *moving (flat-ways, as it were) so that the Plane described by the Axis* OAS *may be perpendicular to that of the Curve.*



Here, putting AP $=x$, PN $=y$, AN $=z$, OA $=d$, OG $=g$, and AG $=a$, the force of the particles in MN will be defined by $2y\times\overline{d+x}|^2$. Therefore the fluent of $2y\dot{x}\times\overline{d+x}|^2$ will be as the whole force of the plane NAM (or AEF, when $x=$ AS) and consequently $C=\frac{flu.\ \overline{d+x}|^2\times y\dot{x}}{flu.\ \overline{d+x}\times y\dot{x}}$: which, therefore, when the point of suspension is in the vertex A, will become $C = \frac{flu.\ yx^2\dot{x}}{flu.\ yx\dot{x}}$. Let this value be denoted by $v$; then, the distance of the centers of gravity and oscillation being $v-a$, we have (by Art. 184) $g : a :: v - a : \left(\frac{a \times \overline{v-a}}{g}\right)$ the distance of the same centers, when the point of suspension is at O, and consequently $C$, in that case, $= g + \frac{a \times \overline{v-a}}{g}$: which form will be found more commodious than the foregoing in most cases.

After the same manner the value of $C$, with respect of the arch A E F, will appear to be $= \frac{flu.\ \overline{d+x}|^2 \times \dot{z}}{flu.\ \overline{d+x} \times \dot{z}}$ $= g + \frac{a \times \overline{v-a}}{g}$, supposing $v = \frac{flu.\ x^2\dot{z}}{flu.\ x\dot{z}}$.

It may not be improper to give an example or two of the use of the foregoing theorems.

191. Let, therefore, E A F be first considered as an isosceles triangle: in which case A P ($x$) and P N ($y$) being in a constant ratio, we have $y = \frac{bx}{c}$ (supposing S F $= b$, and A S $= c$). Hence $C \left(= \frac{flu.\ \overline{d+x}|^2 \times y\dot{x}}{flu.\ \overline{d+x} \times y\dot{x}}\right)$ $= \frac{flu.\ (d^2x\dot{x} + 2dx^2\dot{x} + x^3\dot{x})}{flu.\ (dx\dot{x} + x^2\dot{x})} = \frac{\frac{1}{2} d^2 + \frac{2}{3} dx + \frac{1}{4} x^2}{\frac{1}{2} d + \frac{1}{3} x} =$ $\frac{6d^2 + 8dx + 3x^2}{6d + 4x}$: or (according to the second form) because $v \left(\frac{flu.\ yx^2\dot{x}}{flu.\ yx\dot{x}}\right) = \frac{3x}{4}$, and $a$ is known to

be $= \frac{2x}{3}$,* we have $C \left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{x^2}{18g}$, where $g\ (= d+a) = d + \frac{2}{3}x$. * Art. 175.

Again, because $\dot{z}$ and $\dot{x}$ are in a constant ratio, we also have $\frac{flu.\ \overline{d+x}|^2 \times \dot{z}}{flu.\ \overline{d+x} \times \dot{z}} = \frac{flu.\ \overline{d+x}|^2 \times \dot{x}}{flu.\ \overline{d+x} \times \dot{x}} = \frac{d^2 + dx + \frac{1}{3}x^2}{d + \frac{1}{2}x}$; whence the center of oscillation of the lines E H and A F is given.

192. For a second example, let E A F be supposed a parabola of any kind, whose equation is $y = \frac{x^n}{g^{n-1}}$: then (*according to form* 2) we shall first have $v \left(= \frac{flu.\ yx^2\dot{x}}{flu.\ yx\dot{x}}\right) = \frac{flu.\ x^{n+2}\dot{x}}{flu.\ x^{n+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3}$: whence, $a$ being $= \frac{\overline{n+1} \times x}{n+2}$,† we also get $C \left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{\overline{n+1} \times x^2}{\overline{n+2}|^2 \times \overline{n+3} \times g}$; where $g = d + \frac{\overline{n+1} \times x}{n+2}$. † Art. 176.

But, with respect to the arch of the curve, $v \left(= \frac{flu.\ x^2\dot{z}}{flu.\ x\dot{z}}\right)$ is $= \frac{flu.\ x^2\dot{x}\sqrt{c^{2n-2} + n^2x^{2n-2}}}{flu.\ x\dot{x}\sqrt{c^{2n-2} + n^2x^{2n-2}}}$: from which value (found by infinite series, and even without in some cases)‡ that of $C$ will also be given. ‡ Art. 138.

193. Case 6. *Let the proposed Figure be a Curve vibrating (edge-ways) so that the Motion of the Axis may be in the Plane of the Curve.*

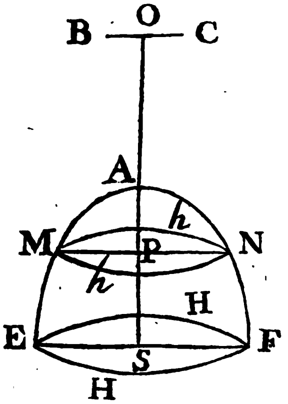
Then (*by Case* 2) the force of all the particles in the line P N (*see the preceding figure*) being defined by $OP^2 \times PN + \frac{1}{3}PN^3$, or $\overline{d+x}|^2 \times y + \frac{1}{3}y^3$ (retaining the notation above) we have $C = \frac{flu. (\overline{d+x}|^2 \times y\dot{x} + \frac{1}{3}y^3\dot{x})}{flu. \overline{d+x} \times y\dot{x}}$: which, when the point of suspension is in the vertex A, will become $\frac{flu. (yx^2\dot{x} + \frac{1}{3}y^3\dot{x})}{flu. yx\dot{x}}$: let this (when found) be denoted by $v$; then, it appears from the preceding case, that the general value of $C$ will *also* be represented by $g + \frac{a \times \overline{v-a}}{g}$.

In the same manner the value of $C$, with respect to the arch E A F, will be expounded by $\frac{flu. \overline{\overline{d+x}|^2 + y^2} \times \dot{z}}{flu. \overline{d+x} \times \dot{z}}$, or by $g + \frac{a \times \overline{v-a}}{g}$, supposing $v = \frac{flu. \overline{x^2 + y^2} \times \dot{z}}{flu. x\dot{z}}$.

194. Example. *Let the Equation of the given Curve be* $y = \frac{x^n}{c^{n-1}}$: then $v\ (= \frac{flu. \overline{yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}}{flu. yx\dot{x}}) = \frac{flu. c^{1-n} x^{n+2} \dot{x} + \frac{1}{3} c^{3-3n} x^{3n} \dot{x}}{flu. c^{1-n} x^{n+1} \dot{x}} = \frac{\overline{n+2} \times x}{n+3} + \frac{\frac{1}{3} c^{2-2n} x^{3n+1} \times \overline{n+2}}{\overline{3n+1} \times x^{n+2}} = \frac{\overline{n+2} \times x}{x+3} + \frac{\overline{n+2} \times c^{2-2n} \times x^{2n-1}}{3 \times \overline{3n+1}}$ $= \frac{n+2}{n+3} \times x + \frac{\overline{n+2}}{3 \times \overline{3n+1}} \times \frac{y^2}{x}$: from which the value of $C$ is also given; and from whence it appears, that if $n$ be expounded by 0, $v$ will become $= \frac{2x}{3} + \frac{2y^2}{3x} = \frac{2}{3} \times \frac{x^2 + y^2}{y}$; in which case the figure will degenerate to a rectangle: but, if $n$ be interpreted by 1, the figure E A F will then be an isosceles

triangle, and $v = \frac{3x}{4} + \frac{y^2}{4x}$: lastly, if $n$ be taken $= \frac{1}{2}$, the curve will be the common parabola, and $v = \frac{5x}{7} + \frac{c}{3}$.

195. Case 7. *Let the Figure* A E F H *be a Solid generated by the Rotation of the Curve* E A F *about its Axis* A S; *having its base* H H *parallel to the Axis of Motion* B O C.



It appears, *from Case* 4, that the force of all the particles in the circular section *hh* (parallel to HH) will be expressed by $\overline{OP^2 + \frac{1}{4}PN^2} \times$ *circle h h*, or $\overline{OP^2 \times PN^2 + \frac{1}{4}PN^4} \times p^2$ ($p$ being = 3.1415, &c.) which, in algebraic terms, is $\overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4}y^4} \times p$ Hence we have $C =$

$$\frac{flu.\ \overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4}y^4} \times p\dot{x}}{flu.\ \overline{d+x} \times py^2\dot{x}}{}^{*} = \frac{flu.(\overline{d+x}|^2 \times y^2\dot{x} + \frac{1}{4}y^4\dot{x})}{flu.\ \overline{d+x} \times y^2\dot{x}}.$$

Which, therefore, when the point of suspension is in the vertex A, becomes $\frac{flu.\ y^2x^2\dot{x} + \frac{1}{4}y^4\dot{x}}{flu.\ y^2x\dot{x}} = v$; and consequently $C = g + \frac{a \times \overline{v-a}}{g}$, as in the preceding cases.

But, with regard to the superficies of the solid, it is found, in Case 4, that the force of the particles in the periphery M $h$ N $h$ is expressed by $\overline{OP^2 + \frac{1}{2}PN^2} \times$ *periph.* $MhNh = \overline{d+x}|^2 \times 2py + py^3$.

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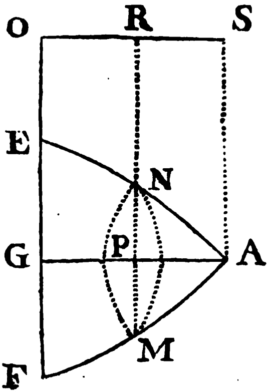
* Art. 185.

Hence the fluent of $\overline{\overline{d+x}|^2 \times 2py+py^3} \times \dot{z}$, divided by that of $\overline{d+x} \times 2py\dot{z}$ $\left(= \frac{flu.\ (\overline{d+x}|^2 \times 2y\dot{z}+y^3\dot{z})}{flu.\ \overline{d+x} \times 2y\dot{z}}\right)$ will give the true value of $C$ with respect to the curve surface E*h*A*h*F. Which, putting $v = \frac{flu.\ (2yx^2\dot{z}+y^3\dot{z})}{flu.\ 2yx\dot{z}}$, is likewise expressed by $g + \frac{a \times \overline{v-a}}{g}$.

196. Ex. 1. *Let* E A F *be considered as a Cone;* then, putting A S $=f$, S F $=b$, and A F $=c$, we have $y = \frac{bx}{f}$ $z = \frac{cx}{f}$; and therefore $C$ $\left(= \frac{flu.\ (\overline{d+x}|^2 \times y^2\dot{x}+\frac{1}{4}y^4\dot{x})}{flu.\ \overline{d+x} \times y^2\dot{x}}\right)$ $= \frac{20d^2 + 30fd + 12f^2 + 3b^2}{20d + 15f}$, when $x = f$. But, with respect to the convex superficies, $C$ will be found $=$ $\frac{12d^2 + 16df + 6f^2 + 3b^2}{12d + 8f}$

197. Ex. 2. *Let* E A F, *&c. be considered as a Sphere whose Center is* S, *and Radius* A S $=r$; in which case $y^2$ being $= 2rx - x^2$, we have $v$ $\left(= \frac{flu.\ \overline{y^2x^2\dot{x} + \frac{1}{4}y^4\dot{x}}}{flu.\ y^2x\dot{x}}\right)$ $= \frac{flu.\ (r^2x^2\dot{x} + rx^3\dot{x} - \frac{3}{4}x^4\dot{x})}{flu.\ (2rx^2\dot{x} - x^3\dot{x})} = \frac{\frac{1}{3}r^2 + \frac{1}{4}rx - \frac{3}{20}x^2}{\frac{2}{3}r - \frac{1}{4}x}$ whence $C$ is also given. But, when $x = 2r$ (or the whole sphere is taken) $v = \frac{7r}{5}$: therefore $a$ being $= r$, and $g =$ O S, in this case, we have $C$ $\left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{r \times 2r}{5g} = g + \frac{2r^2}{5g}$.



198. Case 8. *Let the Figure proposed be a Solid, as in the preceding Case, but let its Axis* A G *be here parallel to the Axis of Motion* O R S.

Then, if R P (O G) be put $= g$, 3,1459, &c. $= p$, A P $= x$, &c. the force of the particles in the circle N M (parallel to E F) will be exhibited by $\overline{g^2 + \frac{1}{2}y^2} \times py^2$, or $pg^2y^2 + \frac{1}{2}py^4$ (*vide Case* 3). Hence we have $C =$ $\frac{flu.\ (pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x})}{g \times solid}$* $= \frac{flu.\ (pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x})}{g \times flu.\ py^2\dot{x}}$† $=$

$$g + \frac{flu.\ \frac{1}{2}y^4\dot{x}}{g \times flu,\ y^2\dot{x}}.$$

Moreover, with respect to the superficies, the force of the particles in the periphery of the said circle M N being $2pg^2y + 2py^3$,‡ we have, in this case, $C =$

$$\frac{flu.\ \overline{2pg^2y + 2py^3} \times \dot{z}}{g \times superficies} = \frac{flu.\ (2pg^2y\dot{z} + 2py^3\dot{z})}{g \times flu.\ 2py\dot{z}} = g + \frac{flu.\ y^3\dot{z}}{g \times flu. y\dot{z}}.$$

199. Ex. 1. *Let* E A F *be a Segment of a Sphere, whose Radius is r;* then $y^2$ being $= 2rx - x^2$, we shall have

$$C\ (g + \frac{flu.\ \frac{1}{2}y^4\dot{x}}{g \times flu. y^2\dot{x}}) = g + \frac{flu.\ (2r^2x^2\dot{x} - 2rx^3\dot{x} + \frac{1}{2}x^4\dot{x})}{g \times flu.\ (2rx\dot{x} - x^2\dot{x})}$$

$$= g + \frac{\frac{2}{3}r^2x - \frac{1}{2}rx^2 + \frac{1}{10}x^3}{g \times \overline{r - \frac{1}{3}x}} = g + \frac{\overline{20r^2 - 15rx + 3x^2} \times x}{\overline{30r - 10x} \times g}$$

Which, when $x$ is expounded, either by $r$ or $2r$, becomes $= g + \frac{2r^2}{5g}$, for the true value of $C$, when either the hemisphere, or whole sphere, is taken. But, with respect to the center of oscillation of the superficies thereof, we have $\dot{z}$ in this case $= \frac{r\dot{x}}{\sqrt{2rx - x^2}}$ * * Art. 142. $= \frac{r\dot{x}}{y}$: and therefore $g + \frac{\text{flu. } y^3\dot{z}}{g \times \text{flu. } y\dot{z}} = g + \frac{\text{flu. } \overline{2rx - x^2} \times r\dot{x}}{g \times \text{flu. } r\dot{x}} = g + \frac{rx - \frac{1}{3}x^2}{g}$: which, when $x = r$, or $x = 2r$, becomes $g + \frac{2r^2}{3g}$.

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* Art. 185.

† Art. 145.

‡ Art. 185.

200. Ex. 2. *Let the Solid* EAF *be a Paraboloid, whose generating Curve is defined by the Equation* $y = \frac{x^n}{c^{n-1}}$: then $C = g + \frac{\text{flu. } \frac{1}{2}y^4\dot{x}}{g \times \text{flu. } y^2\dot{x}} = g + \frac{\text{flu. } \frac{1}{2}x^{4n}\dot{x} \times c^{4-4n}}{g \times \text{flu. } x^{2n}\dot{x} \times c^{2-2n}}$ $= g + \frac{\overline{2n+1} \times x^{2n}}{\overline{4n+1} \times 2g \times c^{2n-2}} = g + \frac{\overline{2n+1} \times y^2}{\overline{4n+1} \times 2g}$. Where if $n$ be taken $= 0$, the figure will become a cylinder, and $C = g + \frac{y^2}{2g}$: but if $n$ be expounded by 1, the figure will be a cone, and $C = g + \frac{3y^2}{10g}$. Lastly, if $n$ be taken $= \frac{1}{2}$, the figure will be the solid generated from the common parabola and $C = g + \frac{y^2}{3g}$.

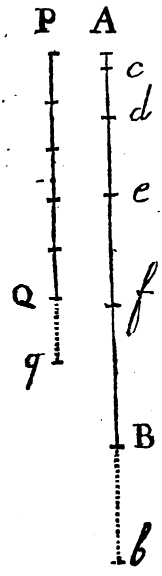
# SECTION XII.

## *Of the Use of Fluxions in determining the Motion of Bodies affected by Centripetal Forces.*

### PROPOSITION I.

201. *THE motion, or velocity, acquired by a body freely descending from rest, by the force of an uniform gravity, is proportional to the time of its descent; and the space gone over, as the square of that time.*

The first part of the proposition is almost self-evident: for, since any motion is proportional to the force by which it is generated, that generated by the force of an uniform gravity must be as the time of descent; because the whole effect of such a force, acting equally every instant, is as that time.



Let, now, the velocity acquired during a descent of one second of time, be such as would carry the body uniformly over any distance $b$ in one second; and let AB $(x)$ denote the distance descended in any proposed time $t$; which time let be denoted by PQ; making $Bb = \dot{x}$ and $Qq = \dot{t}$: then it will be, as $1'' : t :: b : (bt)$ the distance that would be uniformly described in $1''$, with the velocity at B: also $1'' : \dot{t} ::$ the said distance $(bt)$ to $bt\dot{t} = \dot{x}$.* By taking the fluent whereof we get

Art. 2.

$\frac{1}{2}bt^2=x$. Therefore the distance descended ($\frac{1}{2}bt^2$) is as the square of the time. *Q. E. D.*

*Otherwise, without Fluxions.*

Conceive the time (P Q) of falling through A B to be divided into an indefinite number of very small equal particles, represented each by $m$; and let the distance descended in the first of them be A $c$, in the second $cd$, in the third $de$, &c. &c. Then, the velocity being always as the time from the beginning of the descent, it will in the middle of the first of the said particles be defined by $\frac{1}{2}m$; in the middle of the second by $1\frac{1}{2}m$; in the middle of the third by $2\frac{1}{2}m$, &c. &c. But, since the velocity at the middle of any particle of time, is a mean between those at the two extremes, or betwixt any other two equally remote from it, the corresponding particle of the distance A B may, therefore, be considered as described by that mean velocity. And so, the spaces $Ac$, $cd$, $de$, $ef$, &c. described in equal times, being respectively as the said mean celerities $\frac{1}{2}m$, $1\frac{1}{2}m$, $2\frac{1}{2}m$, $3\frac{1}{2}m$, &c. it follows, by addition, that the distances A$c$, A$d$, A$e$, A$f$, &c. gone over from the beginning, are to one another as $\frac{m}{2}, \frac{4m}{2}, \frac{9m}{2}, \frac{16m}{2}$ &c. or 1, 4, 9, 16, 25, &c. that is, as the squares of the times. *Q. E. D.*

### COROLLARY 1.

**202.** Since the distance that might be uniformly run over in one second, with the velocity at B, is expressed by $bt$, the distance that might be described with the same velocity in the time $t$ will therefore be expressed by $bt \times t$, or $bt^2$: whence it appears, that the space A B ($\frac{1}{2}bt^2$) through which the body falls in any given time $t$, is just the half of that which would be uniformly described with the celerity at B, in the same time.

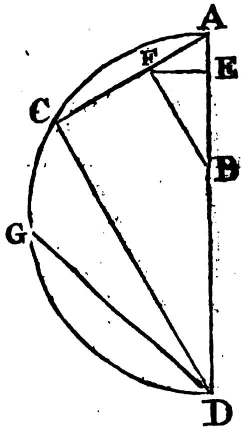
Therefore, since it is found from experiment, that a body near the earth's surface (where the gravity may be taken as uniform) descends about $16\frac{1}{12}$ feet in the first second, it follows that the value of $b$ (is in this case) $=2\times 16\frac{1}{12}=32\frac{1}{6}$: and consequently the number of feet descended in $t$ seconds, equal to $16\frac{1}{12}\times t^2$.

### Corollary 2.

203. It is evident, whatever force the body descends by, the value of $b$ will always be as that force; since a double force, in the same time, generates a double velocity; a treble force, a treble velocity, &c. Therefore, seeing our equation $\frac{1}{2}bt^2=x$, also gives $t=\sqrt{\frac{x}{\frac{1}{2}b}}$, and $b=\frac{x}{\frac{1}{2}t^2}$, it follows,

1. That the distance descended is, universally, as the force and the square of the time conjunctly.
2. That the time is always as the square root of the distance applied to the force.
3. And that the force is as the distance applied to the square of the time.—And it may be further observed, that whatever is here said with regard to the time, also holds in the velocity, being proportional to the time.

## PROPOSITION II.



204. *To determine the Velocity and Time of Descent of a Body along an inclined Plane* A C.

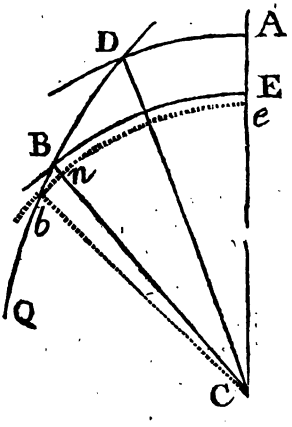
From any point F, in A C, draw FE perpendicular to the vertical line A D, and make FB and CD perpendicular to A C, meeting A D in B and D. Because (by the principles of mechanics) the force of gravity in the direction F C, whereby the body is made to descend along the plane, is to the absolute force thereof, as A F to

AB, or as AC to AD; and since (*by Case* 1. *Art.* 203) the distances descended in equal times are as the forces, it follows that the time of descent through A F will be equal to the time of the perpendicular descent through A B: and consequently the time of descent through AC equal to that through AD; which is given by Prop. 1. Moreover, because the velocities at F and B, acquired in equal times, are as the forces, or as A F to AB; and it appears from Prop. 1, that the velocity at E is to that at B, as $\sqrt{AE} : \sqrt{AB}$, or as $\sqrt{AE \times AB}\ (=AF) : \sqrt{AB \times AB}\ (=AB)$ it follows, by equality, that the celerity at F is equal to that at E; which is therefore given by the preceding proposition. *Q. E. I.*

COROLLARY.

205. Hence the time of descent along the chord AC of a semi-circle ACD is equal to the time of descent along the vertical diameter A D: and, if the chord D G be of the same length with A C (its inclination to the horizon being also the same) the time of descent along it will also be equal to that along the vertical diameter.

## PROPOSITION III.



206. *If, from two Points* A *and* D, *equally remote from the Center of Attraction* C, *two Bodies move with equal Celerities, the one along the Right-line* A C, *the other in a Curve-line* DBQ, *their Celerities, at all other equal Distances from the Center, will be equal.*

For, let CB and CE be any two such distances; let the arch BE be described, from the center C, and also $eb$, indefinitely near to it, cutting C B in $n$: let the centripetal force at the distance of C B or C E be represented by $f$, and the velocity at B, by $v$.

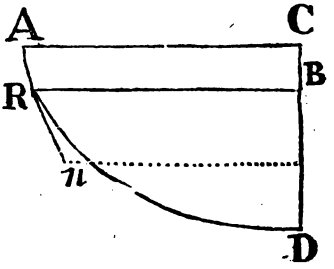
By the resolution of forces, the efficacy of the force ($f$) in the direction $Bb$, whereby the velocity of the body is accelerated, will be $\frac{Bn}{Bb} \times f$: also the time of moving over $Bb$ (being as the distance applied to the velocity) is represented by $\frac{Bb}{v}$: therefore the increase of velocity, in moving through $Bb$, being as the force and time conjunctly, will be defined by $\frac{Bn}{Bb} \times f \times \frac{Bb}{v}$, or its equal $\frac{Bn}{v} \times f$. In the same manner, the velocity at E being denoted by $w$, the time of falling through $Ee$ will be represented by $\frac{Ee}{w}$, and the velocity generated in that time by $\frac{Ee}{w} \times f$: which is to that $\left(\frac{Bn}{v} \times f\right)$ acquired in falling through the arch $Bb$, as $\frac{1}{w}$ to $\frac{1}{v}$. Therefore, seeing the corresponding increments of velocity are always reciprocally as the velocities themselves, it is manifest, if those velocities are equal, in any two corresponding positions of the bodies, they will be so in all others, being always increased alike. But they are equal at A and D by supposition: Therefore, &c. *Q. E. D.*

## PROPOSITION IV.

207. *To find the Ratio of the Velocities, and Times of Descent of Bodies, in Curves; the Force of Gravity being considered as uniform.*

Let A R D represent a curve of any kind, along which a body descends by the force of its own gravity from A; let A C, R B, &c. be parallel, and C D perpendicular, to the horizon; moreover, let R $n$ touch the Curve at R; and let C B = $u$, A R = $w$, and R $n = \dot{w}$.*

* Art. 135.



Since the points B and R (as well as C and A) may be looked upon as equally remote from the earth's center (to which the gravitation tends), the velocity acquired in descending through the arch A R will *(by the last proposition)* be equal to that acquired by falling freely through the right-line C B; which last velocity (by Prop. 1) is always as $\sqrt{CB}$ (or $u^{\frac{1}{2}}$). Therefore the celerity, whether the body moves in a right-line, or a curve, is always in the subduplicate ratio of the perpendicular descent; and so, the time in which R $n$ *($\dot{w}$)* would be uniformly described, with that celerity, will be universally as $\frac{\dot{w}}{u^{\frac{1}{2}}}$; whose fluent is as the time of falling through A R. *Q. E. I.*

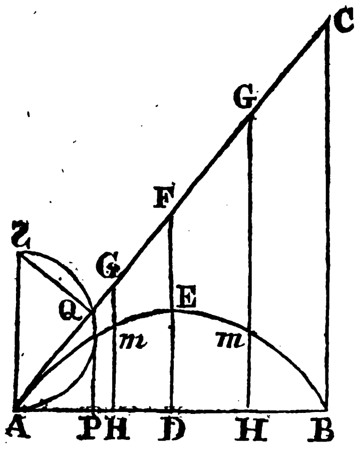
## EXAMPLE.

208. Let the curve A R D be any portion of the common cycloid; whereof the vertex is D and axis D C; and whose nature (putting D C $= c$, and the ray of curvature at D $= a$) is defined by the equation $2a \times DB = DR^2$. Here, we have DR ($= \sqrt{2a} \times \sqrt{DB}$) $= \sqrt{2a} \times \overline{c - u}|^{\frac{1}{2}}$; whose fluxion $- \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\overline{c - u}|^{\frac{1}{2}}}$, with a contrary sign, is the value of R$n$ or $\dot{w}$;

therefore $\frac{\dot{w}}{u^{\frac{1}{2}}} = \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\sqrt{cu - u^2}}$: whose fluent, at the lowest point D, where $u$ becomes $= c$, will (by Art. 142) be equal to $\sqrt{2a}$ multiplied by $\left(\frac{3.14159 \text{ \&c.}}{2}\right)$ half the measure of the periphery of the circle whose diameter is unity. Which fluent (and consequently the time of descent) will therefore continue the same, let the arch D A be what it will.

## PROPOSITION V.

209. *To determine the Paths of Projectiles near the Earth's Surface (neglecting the Resistance of the Atmosphere).*



Let a body be projected from the point A, in the direction of the line A C, with a velocity sufficient to carry it uniformly over the distance $d$ in the time $t$; and let the space through which it would freely descend, by its own gravity, in that time, be denoted by $b$; also let the sine of the angle of elevation B A C (to the radius $r$) be put $= s$, its co-sine $= c$, and the distance of the point A from the ordinate H $m$ (considered as moving parallel to itself along with the body) $= x$; then, *by Trig.* H G (perpendicular to AB) will be $= \frac{sx}{c}$, and A G $= \frac{rx}{c}$.

Because the projectile is turned aside, continually, from a rectilinear path, by the earth's attraction, it must describe a curve-line A$m$ E $m$B, to which AC is a tangent at the point A: but that attraction, acting always in a direction ($m$ H) perpendicular to the horizon, can have no effect upon that part of the velocity with which the body approaches the line B C, parallel to H $m$; therefore the right-line H G (in which the body is always found) will continue to move uniformly towards B C, the same as if gravity was not to act; and the distance G $m$ descended from the tangent A C, by means of the attraction, will be the very same as if the body was to descend from rest along the line G H. This being premised, it is evident, that as $d$ : A G $\left(\frac{rx}{c}\right) :: t : \left(\frac{rx}{cd} \times t\right)$ the time of describing A $m$; and, as $t^2 : \frac{r^2x^2}{c^2d^2} \times t^2 :: b : \frac{br^2x^2}{c^2d^2}$ the space (G $m$) through which a body would freely descend in that time (by Prop. 1).

Hence $\frac{sx}{c} - \frac{br^2x^2}{c^2d^2}$, or $\frac{csd^2x - br^2x^2}{c^2d^2}$ is a general value for the ordinate $m$ H: by putting which $= 0$, we get $x = \frac{csd^2}{br^2}$ = A B = the amplitude of the projection. But, by putting its fluxion equal to nothing, we have $x = \frac{csd^2}{2br^2}$; which substituted for $x$ in the value of $m$ N, gives $\frac{s^2d^2}{4br^2}$ for the altitude D E of the projection. *Q. E. I.*

## COROLLARY.

210. If another body be projected, with the same celerity, in the vertical direction A S; then, $s$ becoming $= r$, the altitude of that projection $\left(\frac{s^2d^2}{4br^2}\right)$ will be-

come $\frac{d^2}{4b} = AS$; which call $h$, and let this value be substituted in those of A B and D E, and they will become $\frac{4hcs}{r^2}$ and $\frac{hs^2}{r^2}$ respectively.

Hence, if from the point Q where the line of direction A C cuts a semi-circle described upon A S, the lines S Q and Q P be drawn, the latter perpendicular to A B, the triangles A S Q and A Q P being similar, we shall have

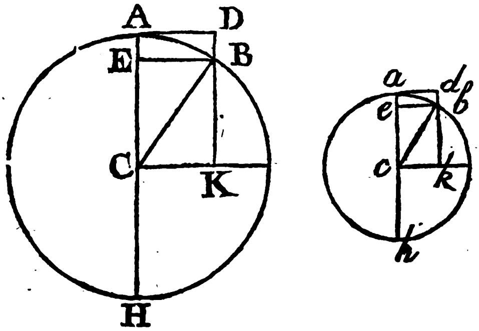
$$r : s :: h\ (AS) : \frac{sh}{r} = AQ$$

$$r : s :: \frac{sh}{r}\ (AQ) : \frac{s^2h}{r^2} = PQ = DE$$

$$r : c :: \frac{sh}{r}\ (AQ) : \frac{sch}{r^2} = AP = \tfrac{1}{4} AB$$

## PROPOSITION VI.

211. *To determine the Ratio of the Forces, whereby Bodies, tending to the Centers of given Circles, are made to revolve in the Peripheries thereof.*



Let A B H and $a\,b\,h$ be any two proposed circles, whereof let A B and $a\,b$ be similar arcs; in which, let the velocities of the revolving bodies be respectively as $V$ to $v$; make DBK and *dbk* parallel to the radii AC and *ac*, putting $AC=R$, $ac=r$, and the ratio of the centripetal force in A B H to that in *a b h*, as $F$ to $f$.

It is plain, because the angles A B D and *a b d* are equal, that the velocities at B and *b*, in the directions B K and *b k*, with which the bodies recede from the tangents A D and *a d*, are to each other as the absolute celerities $V$ and $v$.* But those velocities, being the effects of the centripetal forces acting in corresponding similar directions during the times of describing A B and *a b*, will therefore be as the forces themselves when the times are equal; but when unequal, as the forces and times conjunctly. Therefore, the times being universally as $\frac{AB}{V}$ to $\frac{ab}{v}$, or as $\frac{R}{V}$ to $\frac{r}{v}$ (because the arcs AB and *ab* are similar) we have, as $F\times\frac{R}{V} : f \times \frac{r}{v} :: V : v$; whence (multiplying the antecedents by $\frac{V}{R}$ and the consequents by $\frac{v}{r}$) it will be, as $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$: therefore the forces are as the squares of the velocities directly, and as the radii inversely. * Art. 35.

*Otherwise.*

Let the indefinitely little arch A B be the distance that the body moves over in a given, or constant particle of time; and let the centripetal force at B be measured by twice the subtense or space A E through which the body is drawn from the tangent A D in that time.†

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† The velocity which any force, uniformly continued, is capable of generating, in a given body, in a given time, is the proper measure of the intensity of that force.* But this velocity is itself measured by the space the body would move uniformly over in a given time; which space is always the * Art. 203.

Then by the nature of the circle, $AB^2 = AH \times AE = AC \times 2AE$, and consequently $2AE = \frac{AB^2}{AC}$: therefore, the force is as the square of the velocity applied to the radius of the circle *(as before)*.

### Corollary I.

212. Because, $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$, it follows that

$$V : v :: \sqrt{RF} : \sqrt{rf}, \text{ and}$$
$$R : r :: \frac{V^2}{F} : \frac{v^2}{f}.$$

### Corollary II.

213. If the ratio of the periodic times be denoted by that of $P$ to $p$; then the ratio of the velocities $V$, $v$ being as $\frac{R}{P}$ to $\frac{r}{p}$, we shall have, by equality, $\sqrt{RF} : \sqrt{rf} :: \frac{R}{P} : \frac{r}{p}$; whence also

$$F : f :: \frac{R}{P^2} : \frac{r}{p^2}, \text{ and}$$
$$R : r :: FP^2 : fp^2.$$

---

double of that through which the body would freely descend, from rest, in the same time.* Therefore 2AE is the proper measure of the centripetal force, according as we have assumed it.—It is true, when the forces to be compared are all computed in the same manner, from the nascent, or indefinitely small subtenses of contemporaneous arcs, it matters not whether we consider those subtenses, or their doubles, as the measures of the forces, the ratio being the same in both cases. But when the forces so found are to be compared with others derived from a fluxional calculus, it is absolutely necessary to take the double subtense for the measure of the force.—This Note is inserted, that the learner may avoid the errors, which some very considerable mathematicians have fallen into by not properly attending to this particular.

* Art. 202.

## COROLLARY III.

214. If the measure of the force, or the velocity that might be uniformly generated in a given time (1) be expounded by any power $a^n$ of the radius A C $(a)$; then the distance through which a body would freely descend in the same time, by that force, uniformly continued, will be expressed by $\frac{1}{2}\, a^n$.* Therefore, the distances descended, by means of the same force, uniformly continued, being as the squares of the times,† it is evident, if the time of moving through A B be denoted by $t$, that the distance A E descended in that time, will be denoted by $\frac{t^2}{1^2} \times \frac{1}{2}\, a^n$: and so we shall have A B $(\sqrt{2\text{AE} \times \text{AC}}) = \frac{t}{1} \times a^{\frac{n+1}{2}}$; which being the distance described by the revolving body in the time $t$, it follows that the space gone over in the given time (1) will be $a^{\frac{n+1}{2}}$: which, therefore, is the true measure of the celerity in this case. The same conclusion might have been derived in much fewer words from Corol. 1; but, as a thorough understanding hereof is absolutely necessary in what follows hereafter, I have endeavoured to make it as plain as possible.

## COROLLARY IV.

215. Hence the time of revolution is also derived; for it will be as $a^{\frac{n+1}{2}}$ : 3.14159 &c. × $2a$ (the whole periphery) :: 1 : $\frac{3.14\ \&c. \times 2a}{a^{\frac{n+1}{2}}}$ or 3.14159 &c. × $2a^{\frac{1-n}{2}}$, the true measure of the periodic time.

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* Art. 202.

† Art. 201.

### COROLLARY V.

216. Therefore, if $n$ be expounded by 1, 0, $-1$, $-2$ and $-3$ successively, then the velocity corresponding will be as $a$, $a^{\frac{1}{2}}$, 1, $a^{-\frac{1}{2}}$, and $a^{-1}$; and the time of revolution, as 1, $a^{\frac{1}{2}}$, $a$, $a^{\frac{3}{2}}$ and $a^2$ re-respectively.

### SCHOLIUM.

217. From the preceding proposition, and its subsequent corollaries, *The velocity and periodic time of a body revolving in a circle at any given distance from the earth's center, by means of its own gravity, may be deduced:* for let $d$ be put for the space through which a heavy body, at the surface of the earth, descends in the first second of time, then $2d$ will be the measure of the force of gravity at the surface: and therefore the radius of the earth being denoted by $r$, the velocity, *per* second, in a circle at its surface, will be $\sqrt{2rd}$; and the time of revolution $= \dfrac{3.14159 \text{ \&c.} \times 2r}{\sqrt{2rd}}$ $= 3.14159 \text{ \&c.} \times \sqrt{\dfrac{2r}{d}}$ *(seconds)*; which two expressions, because $r$ is $= 21000000$ feet and $d = 16\frac{1}{12}$ will therefore be nearly equal to 26000 feet and 5075 *seconds*, respectively. Let $R$ be now put for the radius of any other circle described by a projectile about the earth's center: then, because the force of gravitation above the surface is known to vary according to the square of the distance inversely, we have (by Case 4, Corol. 5) $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} :: (26000^{\text{F}})$ the velocity *(per* second) at the surface, to $26000 \times \sqrt{\dfrac{r}{R}}$ the ve-

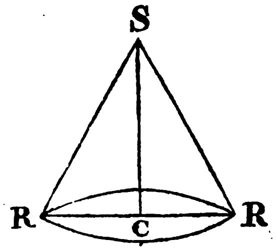
locity in the circle whose radius is $R$: and $r^{\frac{3}{2}} : R^{\frac{3}{2}} ::$ ($5075^{s}$) the periodic time at the surface : to $5075 \times \sqrt{\frac{R^3}{r^3}}$, the time of revolution in the circle $R$. Which, if $R$ be assumed equal to ($60r$) the distance of the moon from the earth, will give $2360000^{s}$, or $27.3^{D}$ nearly, for the periodic time at that distance.

In like sort *the ratio of the forces of gravitation of the moon, towards the sun and earth, may be computed.* For the centrifugal forces in circles, being universally as the radii applied to the squares of the times of revolution, it will be as $\left(\frac{81000000}{1}\right)$ the semi-diameter of the *Magnus Orbis* divided by the square of one year (the periodic time of the earth and moon about the sun) is to ($240000 \times 178$) the distance of the moon from the earth divided by $\frac{1}{178}$, the square of the periodic time of the moon about the earth, so is 1, 9 to 1 nearly; and so is the gravitation of the moon towards the sun to her gravitation towards the earth.

Also, after the same manner, *the centrifugal force of a body at the equator, arising from the earth's rotation, is derived.* For since it is found above, that 5075 seconds is the time of revolution, when the centrifugal force would become equal to the gravity, and it appears (by Case 2, Corol. 2) that the forces, in circles having the same radii, are inversely as the squares of the periodic times, we therefore have, as $\overline{80160}|^{2}$ (the square of the number of seconds in ($23^{H}$ $56^{M}$) one whole rotation of the earth) to $\overline{5075}|^{2}$ (the square of the number of seconds above given) so is the force of gravity (which we will denote by unity) to $\frac{1}{289}$, the centrifugal force of a body at the equator arising from the earth's rotation.

But, to determine, in a more general manner, the ratio of the force of a body revolving in any given circle, to its gravity, we have already given 3. 14 &c. $\times \sqrt{\frac{2r}{d}}$ for the time of revolution at the surface of the earth, when the gravity and centrifugal force are equal: therefore, if the time of revolution in any circle whose radius is $a$, be denoted by $t$, it follows, from Corol. 2, last Prop. that, $\frac{r}{\overline{3.14}|^2 \&c. \times \frac{2r}{d}} : \frac{a}{t^2}$ :: the gravity of the body : to its centrifugal force in that circle; which, therefore, is as unity to $\frac{\overline{3.14}|^2 \&c. \times 2a}{dt^2}$; or as 1 to $1.228 \times \frac{a}{t^2}$ very nearly; where $a$ denotes the number of feet in the radius of the proposed circle, and $t$ the number of seconds in one entire revolution. So that, if the length of a sling, by which a stone is whirled about, be two feet, and the time of revolution $\frac{1}{2}$ of a second, the force by which the stone endeavours to fly off, will be to its weight as 9. 824 to unity.

From this general proportion, *the centrifugal force and periodic time of a pendulum describing a conical surface* may likewise be deduced.



For let S R, the length of the pendulum, be denoted by $g$; the altitude C S of the cone, by $c$; the semi-diameter C R of the base by $a$; and the time of revolution by $t$: then, the force of gravity being represented by unity, the force with which the revolving body at R, the end of the pendulum, tends to recede from the center C, will be defined by $\frac{\overline{3.14\,\&c.}|^2 \times 2a}{dt^2}$, as has been already shown. Therefore, because the body is retained in the circle R R by the action of three different powers, *i. e* the centrifugal force $\left(\frac{\overline{3.14\,\&c.}|^2 \times 2a}{dt^2}\right)$ in the direction C R, the force of gravity (1) in a direction parallel to S C, and the force of the thread or wire R S, compounded of the former two; it follows, from the principles of Mechanics, that as SC $(c)$ to C R $(g)$, so is the weight of the body at R, to the force with which it acts upon the thread or wire R S; and as $1 : \frac{\overline{3.14\,\&c.}|^2 \times 2a}{dt^2}$ $:: \text{C S } (c) : \text{C R } (a)$ : whence $dt^2 = \overline{3.14\,\&c.}|^2 \times 2c$, and $t = 3.14\,\&c. \times \sqrt{\frac{2c}{d}} = 1{,}108\sqrt{c}$ nearly. Because $dt^2$, or its equal $\overline{3.14\,\&c.}|^2 \times 2c$, expresses the space a heavy body will descend, by its own gravity, in the time $t$,* and since $1^2 : \overline{3.14\,\&c.}|^2 :: 2c : \overline{3.14\,\&c.}|^2 \times 2c$ $(= dt^2)$ it therefore appears that, *as the square of the diameter of any circle, is to the square of its periphery, so is twice the perpendicular altitude of the cone, to the distance a heavy body will freely descend in the time of one whole gyration of the pendulum, let the base of the cone, and the length of the pendulum be what they will.*

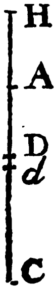
## PROPOSITION VII.

218. *To determine the Ratio of the Velocities of Bodies descending, or ascending, in Right-lines, when accelerated, or retarded, by Forces, varying according to a given Law.*

Suppose a body to move in the right-line C H, and let the force whereby it is urged towards C, or H,

* Art. 202.

be as any variable quantity $F$: moreover, let the velocity of the body be represented by $v$; putting its distance C D, from the point $C = x$, and $D\,d = \dot{x}$.



Then since the time wherein the space $D\,d$ $(\dot{x})$ would be uniformly described, with the velocity at D, is known to be as $\frac{\dot{x}}{v}$, the velocity that would be uniformly generated, or destroyed, in that time by the force $F$ (being as the time and force conjunctly) will consequently be as $\frac{F\dot{x}}{v}$: which therefore must be equal to, $\pm \dot{v}$, the uniform increase or decrease of celerity in that time; and consequently $\pm v\dot{v} = F\dot{x}$. From whence, when the value of $F$ is given in terms of $x$, or $v$, the value of $v$ will likewise be known. *Q. E. I.*

## COROLLARY I.

219. Hence, the law of the velocity being given, that of the force is deduced: for, since $F\dot{x} = \pm v\dot{v}$, it is evident that $F = \pm \frac{v\dot{v}}{\dot{x}}$.

## COROLLARY II.

220. Hence, also, the ratio of the velocity at D to that whereby a body might revolve in the periphery of a circle about the center C, at the distance of C D, will be known: for, if this last velocity be denoted by $w$, the value of $F$ will be rightly expressed by $\frac{w^2}{x}$ *: whence, by substitution, we have $\pm v\dot{v} = \frac{w^2\dot{x}}{x}$, or

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* Art. 212.

$\pm v^2 \times \frac{\dot{v}}{v} = w^2 \times \frac{\dot{x}}{x}$: whence $w^2 : v^2 : \pm \frac{\dot{v}}{v} : \frac{\dot{x}}{x}$, and consequently $w : v :: \sqrt{\frac{\pm \dot{v}}{v}} : \sqrt{\frac{\dot{x}}{x}}$. Where, as well as above, the sign of $\dot{v}$ must be taken + or − according as the body is urged from, or towards the center C.

## PROPOSITION VIII.

221. *Supposing a Body, let go from a given Point* A *with a given Celerity (c) along a Right-line* C H, *to be urged, either way, in that Line, by a Force varying as any Power (n) of the distance from a given Point* C; *to find, not only, the Relation of the Velocities, and Spaces gone over, but also the times of Ascent and Descent.*

The construction of the preceding problem being retained, *F* will here be expounded by $x^n$, and we shall therefore have $\pm v\dot{v}$ $(= F\dot{x}) = x^n \dot{x}$; and consequently, by taking the fluent thereof, $\pm \frac{v^2}{2} = \frac{x^{n+1}}{n+1}$; but to correct the fluent thus found, let $x$ be taken = C A (which we will call $a$) then $v$ being $= c$, the fluent in that circumstance will become $\pm \frac{c^2}{2} = \frac{a^{n+1}}{n+1}$: therefore the fluent duly corrected is $\pm \frac{v^2}{2} \mp \frac{c^2}{2} = \frac{x^{n+1} - a^{n+1}}{n+1}$,* or $v^2 \backsim c^2 = \frac{2x^{n+1} \backsim 2a^{n+1}}{n+1}$: whence $v$ will come out $= \sqrt{c^2 + \frac{\mp 2a^{n+1} \pm 2x^{n+1}}{n+1}}$: where the signs of $v$ and $x^{n+1}$ must be alike, when both quantities increase, or decrease, at the same time; that is,

* Art. 78.

* Art. 220. when the force, from C, is a repulsive one;* but, unlike, when one increases while the other decreases, or the force, tending to C, is an attractive one. In the former case we therefore have $v = \sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}$; and, in the latter, $v = \sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$.

The value of $v$ being thus obtained, let the required time of moving over the space A D be now denoted by T; then, since $\dot{T}$ is universally $= \frac{\dot{x}}{v}$, we have $\dot{T}$

$$= \frac{\dot{x}}{\sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}}, \text{ or } \dot{T} =$$

$$\frac{\dot{x}}{\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}}$$ according to the two foresaid cases respectively: from whence, by finding the fluent, the time itself will be known. *Q. E. I.*

## COROLLARY.

222. If the body proceeds from rest at A, $c$ will be $= 0$, and we shall have $\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2x^{n+1} - 2a^{n+1}}}$, or

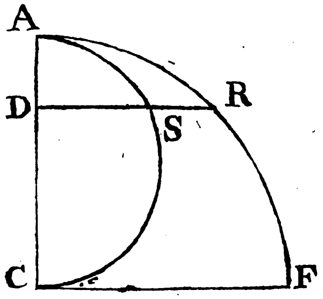
$$\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2a^{n+1} - 2x^{n+1}}}.$$

## SCHOLIUM.

223. Although, the fluents of the expressions given above cannot be exhibited, in a general manner, neither, in finite terms, nor by means of circular arcs and logarithms; yet, in some of the most useful

cases that occur in nature, they may be obtained with great facility.

Thus, if in $\frac{\overline{1+n}|^{\frac{1}{2}}\dot{x}}{\sqrt{2a^{n+1}-2x^{n+1}}}$ (expressing the fluxion of the time of descent along A D) $n$ be expounded by 1, 0, – 2, and –3 successively, the fluxion itself will become equal to $\frac{\dot{x}}{\sqrt{a^2-x^2}}$, $\frac{\dot{x}}{\sqrt{2a-2x}}$, $\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-x^2}}$, and $\frac{ax\dot{x}}{\sqrt{a^2-x^2}}$ respectively: whence, if A R F be a quadrant of a circle whose center is C, and A S C a semi-circle whose diameter is A C, and D S R be perpendicular to A C; then it will appear,



1°. That, when $n=1$, and $\dot{T}=\frac{\dot{x}}{\sqrt{a^2-x^2}}$, the velocity $(\sqrt{a^2-x^2})$ at D will be represented by D R, and the fluent sought by $\frac{AR}{AC}$*.

2°. That, when $n=0$, and $\dot{T}=\frac{\dot{x}}{\sqrt{2a-2x}}$, the velocity at D, and the time of descent through AD, will each be defined by $\sqrt{2AD}$.

3° That, when $n=-2$, and $\dot{T}=\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-x^2}}$ the velocity $\left(\frac{\sqrt{ax-x^2}}{x\sqrt{\frac{1}{2}a}}\right)$ will be as $\frac{DS}{CD\sqrt{\frac{1}{2}AC}}$ and the time of descent through AD, as $\sqrt{\frac{1}{2}AC}\times\overline{AS+DS}$.

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* Art. 142.

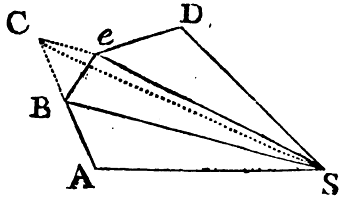
4°. And that, when $n = -3$, and $\dot{T} = \frac{ax\dot{x}}{\sqrt{a^2 - x^2}}$, the velocity will be as $\frac{DR}{AC \times CD}$, and the time as $AC \times DR$.

Hence the time of the whole descent through the radius AC, appears to be as $\frac{AF}{AC}$, $\sqrt{2AC}$, $\sqrt{\frac{1}{2}AC} \times AF$, or $AC^2$. But the time of one whole revolution in the periphery A R F &c. was found to be as $\frac{4AF}{AC^{\frac{n+1}{2}}}$;* which in the four cases above specified is $\frac{4AF}{AC}$, $\frac{4AF}{\sqrt{AC}}$, $4AF \times \sqrt{AC}$, and $4AF \times AC$: therefore, if the time of moving over the quadrant A F be denoted by $Q$, it follows that the time of descent through the radius AC, will be truly defined by $Q$, $Q \times \frac{AC\sqrt{2}}{AF}$, $Q \times \sqrt{\frac{1}{2}}$, or $Q \times \frac{AC}{AF}$ according to the foresaid cases respectively.

* Art. 215.

### LEMMA.

224. *The Areas which a revolving Body describes, by Rays drawn to the Center of Force, are proportional to the Times of their Description.*

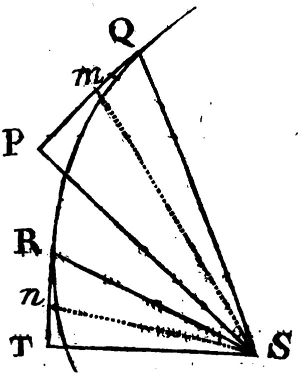


For, let a body, in any given time, describe the right-line A B, with an uninterrupted uniform motion; but upon its arrival at B let it be impelled towards the center S, so that, instead of proceeding along A B C, it may, after the impulse, describe the right-line B *e*.

Because the force, acting in the line S B, can neither add to, nor take from the celerity which the body has in a direction perpendicular to that line, the distance of the body from the said line, at the end of a given time, will therefore be the very same as if no force had acted; and consequently the area B *e* S equal to the area B C S, which would have been described in the same time, had the body proceeded uniformly along B C; because triangles, having the same base and altitude, are equal.

Therefore, seeing no impulse, however great, can affect the quantity of the area described about the center S in a given time, and because the areas A S B, B S C, described about that point when no force acts, are as the bases A B, B C, or the times of their description, the proposition is manifest.

### COROLLARY.



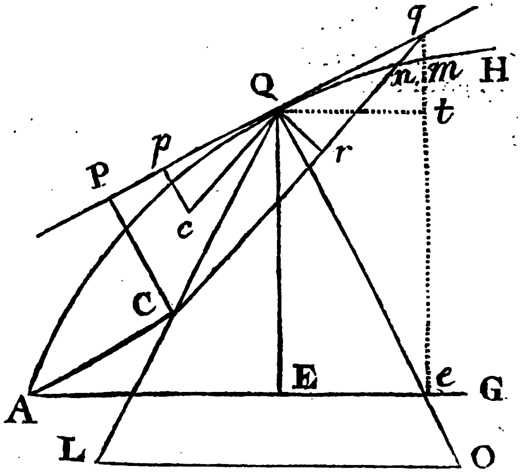
225. *Hence the velocity of a revolving body, at any point* Q *or* R, *is inversely as the perpendicular* S P *or* S T, *falling from the center of force upon the tangent at that point.*

For, let two other bodies *m* and *n* be supposed to move uniformly from Q and R, along the tangents Q P and R T, with velocities respectively equal to those of the revolving body at Q and R; then the distances Q *m* and R *n*, gone over in the same time, will be to each other as those velocities; and the areas Q S *m* and R S *n* will be equal, being equal

to those described by the revolving body in the same
* Art. 113. time :* whence $Qm \times SP$ being $= Rn \times ST$, it follows that $Qm : Rn :: ST : SP :: \frac{1}{SP} : \frac{1}{ST}$.

## PROPOSITION IX.

226. *To determine the Law of the Centripetal Force, tending to a given Point* C, *whereby a Body may describe a given Curve* A Q H.



Let Q P be a tangent to the curve at any point Q; upon which, from the center C, let fall the perpendicular C P; put $CQ = s$, $CP = u$; and let the velocity of the projectile at Q be denoted by $v$.

Therefore, since $v^2$ is always as $\frac{1}{u^2}$ (by the *Corol.* to *Lemma*) it is evident, by taking the fluxions of both quantities, that $v\dot{v}$ will also be as $\frac{-\dot{u}}{u^3}$: but the centripetal force, whether the body moves in a right-line or a curve, is always as $-\frac{v\dot{v}}{\dot{s}}$ (by Art. 219 and 206) Therefore the centripetal force is likewise as $\frac{\dot{u}}{u^3\dot{s}}$. *Q.E.I.*

*The same otherwise.*

227. Let the ray of curvature Q O be denoted by *R*: then, because the centripetal forces in circles are known to be as the squares of the velocities directly and the radii inversely,* it follows that the force tending to the point O, whereby the body might be retained in its orbit at Q, or in the circle whose radius is Q O, will be defined by $\frac{1}{u^2}\times\frac{1}{R}$: whence (by the resolution of forces) it will be C P *(u)* : C Q *(s)* :: $\frac{1}{u^2R}$ (the force in the direction Q O) : $\frac{s}{u^3R}$, the force in the direction Q C: which, because $R=\frac{s\dot{s}}{\dot{u}}$ † will also be expressed by $\frac{\dot{u}}{u^3\dot{s}}$. *Q. E. I.*

* Art. 212.

† Art. 73.

*Another way.*

228. Let *nq* be the indefinitely small part of the right-line C *q*, intercepted by the curve and the tangent Q*q*, expressing the effect of the centripetal force in the time of describing the area Q C *n*. Now these effects, or the distances descended by means of forces uniformly continued, are known to be in the duplicate ratio of the times, ‡ or of the areas denoting those times: § therefore, the centripetal force at Q, or the distance descended by means thereof in a given time, will be as *n q* applied to the second power of the area Q C *q*, or as $\frac{nq}{CP^2\times Qq^2}$. This expression is the same

‡ Art. 201.

§ Art. 224.

with that given by Sir Isaac Newton in his *Principia*, *Book* 1, *Prop.* 6. But, to adapt it to a fluxional *calculus;* let Q E be an ordinate to the principal axis AG; and let (as usual) $AE = x$, $EQ = y$, $AQ = z$, E $e$ (or Q$t$) $= \dot{x}$, Q$q = \dot{z}$; supposing $eq$ (parallel to EQ) to intersect the curve and the tangent in $m$ and $q$.

Since Q$q$ is conceived indefinitely small (or in its nascent state) the triangle $nmq$ may be taken as rectilineal;* also the angle $n =$ CQP and the angle $m =$ Q$qt$: whence it will be (by Trigonometry) as $S.\,CQP\,(n) : S.\,Qqt\,(m) :: mq : nq$; that is, as $\frac{CP}{CQ} : \frac{Qt}{Qq} :: mq : nq = \frac{CQ \times Qt \times mq}{CP \times Qq}$: which substituted above gives $\frac{CQ \times Qt \times mq}{CP^3 \times Qq^3}$ for the measure of the centripetal force at Q: but $mq$ (supposing $x$ to flow uniformly) is known to be as $-\ddot{y}$: therefore the force at Q, is as $\frac{CQ \times Qt \times -\ddot{y}}{CP^3 \times Qq^3}$, or its equal $\frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$; where the divisor $(u^3\dot{z}^3)$ is as the cube of (QC$q$) the fluxion of the area AQC.

* Art. 136.

The very same theorem may likewise be deduced from that given by our second method: for, since $(R)$ the ray of curvature at Q is universally* $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, the value of $\frac{s}{u^3 R}$ (there found) will here, by substitution, become $= \frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$.

* Art. 68.

This expression, though in appearance less simple than $\frac{\dot{u}}{u^3\dot{s}}$, first found, is, for the general part, more commodious in practice.

### COROLLARY I.

229. If the point C be so remote that all right-lines drawn from thence to the curve may be considered as parallel to each other, the force will then (making Q $r$ perpendicular to C $q$) be as $\frac{-s\dot{x}\ddot{y}}{CQ \times \overline{Qr}^3}$, or barely as $\frac{-\dot{x}\ddot{y}}{\overline{Q\,r}^3}$; since $s$ (C Q) in this case may be rejected.

From this expression, which is general, *in all cases where the force acts in the direction of parallel lines, it appears that the force, which always acting in the direction of the ordinate* Q E, *would retain the body in its orbit, is every where as* $\frac{-\ddot{y}}{\dot{x}^2}$; because Q C here coincides with Q E, and Q $r$ becomes $= \dot{x}$.

### COROLLARY II.

230. Because the force, tending to the point C, is universally as $\frac{CQ}{CP^3 \times QO}$ (or $\frac{s}{u^3R}$) the force to any other point $c$, will, by the same argument, be as $\frac{c\,Q}{\overline{cp}^3 \times QO}$. *Hence the forces, to different centers* C *and* c *(about which equal areas are described in the same time) are to each other in the ratio of* $\frac{CP^3}{CQ}$ *to* $\frac{cp^3}{cQ}$ *inversely.*

### COROLLARY III.

231. Moreover, *the ratio of the velocity at* Q *to the velocity whereby the body might revolve in a circle about the center* C, *at the distance* C Q, is easily deduced from hence: for, since the celerity at Q is that

whereby the body might revolve in a circle about the center O, and the forces tending to the centers O and C are to each other as $u$ (C P) and $s$ (C Q); it therefore follows, if the ratio sought be assumed as $v$ to $w$, that $\frac{v^2}{QO} : \frac{w^2}{QC} :: u : s$ (by Art. 212). Whence also $v^2 : w^2 :: u \times QO\ (uR) : s \times QC\ (s^2)$ and consequently $v : w :: \sqrt{\frac{uR}{ss}} : 1 :: \sqrt{\frac{u\dot{s}}{s\dot{u}}} : 1 :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{\frac{\dot{u}}{u}}$ (because $R = \frac{s\dot{s}}{\dot{u}}$).

The same proportion may also be derived from *Corol.* 2. *Prop.* 7. For it is there proved that $v : w :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{-\frac{\dot{v}}{v}}$; and it appears from above, that $-\frac{\dot{v}}{v} = \frac{\dot{u}}{u}$: whence the whole is manifest.

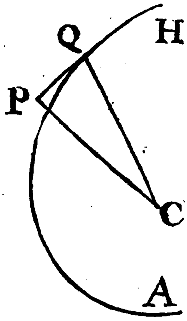
If O L be made perpendicular to Q C, Q L will be $\left(= \frac{CP \times QO}{CQ}\right) = \frac{uR}{s}$, and $\frac{QL}{CQ} = \frac{uR}{s^2}$; and therefore $v : w :: QL^{\frac{1}{2}} : CQ^{\frac{1}{2}}$: which is another proportion of the proposed celerities.

### COROLLARY IV.

232. Lastly, the law of centripetal force being given, the nature of the trajectory A Q may from hence be found; for since the force $(F)$ is universally defined by $\frac{\dot{u}}{u^3\dot{s}}$, it is evident that $\frac{-1}{2u^2}$ will be = the fluent of $F\dot{s}$; which, when $F$ is given in terms of $s$, will become known; and then, the relation between $u$ and $s$ being given, the curve itself is known.

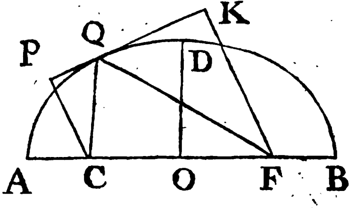
## EXAMPLE I.

233. *Let the given Curve* AQH *be the Logarithmic Spiral, and* C *the Center thereof:* then $u$ (CP) being in this case $= \frac{bs}{a}$,* we have $\frac{\dot{u}}{u^3\dot{s}}$† $\left(=\frac{b\dot{s}}{a\dot{s}} \times \frac{a^3}{b^3s^3}\right) = \frac{a^2}{b^2s^3}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$‡ $\left(= \sqrt{\frac{bs\dot{s}}{a} \times \frac{a}{bs\dot{s}}}\right) =$ unity. Hence, it appears that the force is inversely as the cube of the distance; and the velocity, every where, equal to that whereby the body might revolve in a circle at the same distance.



## EXAMPLE II.

234. *Let it be required to find the Law of the Centripetal Force, whereby a Body, tending to the Focus* C, *is made to revolve in the Periphery of an Ellipsis* AQDB.



From the other focus F draw FK parallel to CP meeting the tangent PQ (at right-angles) in K, join FQ; putting the transverse axis AB $= a$, the semi-conjugate OD $= \frac{1}{2}b$, and the parameter $\left(\frac{b^2}{a}\right) = p$: then, CQ and CP being denoted as above,§ we have FQ $(=\text{AB}-\text{CQ}) = a - s$; whence, by reason of the similar triangles CQP and FQK, it will be

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* Art. 61.

† Art. 227.

‡ Art. 231.

§ Art. 231.

$s : u :: a-s : \mathrm{FK} = \frac{\overline{a-s}\times u}{s}$. But $\mathrm{FK}\times\mathrm{CP}$ is $=\mathrm{OD}^2$ (by the nature of the curve). Hence we get $\frac{\overline{a-s}\times u^2}{s} = \frac{1}{4}b^2$; and consequently $\frac{1}{u^2} = \frac{4a}{b^2 s} - \frac{4}{b^2}$; whereof the fluxion being $-\frac{2\dot{u}}{u^3} = -\frac{4a\dot{s}}{b^2 s^2}$, we obtain

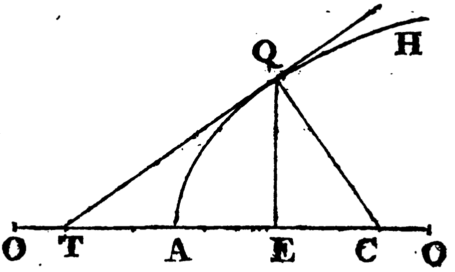
* Art. 227. † Art. 231. $\frac{\dot{u}}{u^3\dot{s}}$ * $= \frac{2a}{b^2}\times\frac{1}{s^2} = \frac{2}{ps^2}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$ † $= \sqrt{\frac{2\times\overline{a-s}}{a}} = \sqrt{\frac{\mathrm{FQ}}{\mathrm{AO}}}$. Hence, it appears that *the centripetal force is, in this case, as the square of the distance inversely; and the velocity at* Q *is to that whereby the body might describe a circle at the distance* C Q, *every where, in the ratio of* $\mathrm{FQ}^{\frac{1}{2}}$ *to* $\mathrm{AO}^{\frac{1}{2}}$.

If the curve had been an hyperbola; then $\frac{a+s}{s}\times u^2$ (instead of $\frac{a-s}{s}\times u^2$) would have been $= \frac{1}{4}b^2$, and so $\frac{\dot{u}}{u^3\dot{s}} = \frac{2a}{b^2}\times\frac{1}{s^2} = \frac{2}{ps^2}$, the very same as before.

But, had it been a parabola, the equation would have been $\frac{a+0}{s}\times u^2 = \frac{1}{4}b^2$, or $\frac{u^2}{s}\ (= \frac{b^2}{4a}) = \frac{1}{4}p$; and the force, *still*, as $\frac{2}{ps^2}$. But the measure of the velocity $\left(\sqrt{\frac{u\dot{s}}{s\dot{u}}} = \sqrt{\frac{2a-2s}{a}}\right)$ in this case becoming barely $= \sqrt{2}$, it follows that *the velocity in a parabola is to that whereby the body might describe a circle at the same distance from the center, in the constant ratio of* $\sqrt{2}$ *to unity.*

## EXAMPLE III.

235. *Let it be required to find the Law of the Centripetal Force, by which a Body, tending to any given Point* C, *in the Axis, is made to describe a conic Section* AQH.



Put the semi-transverse axis (O A) $=a$, the semi-conjugate $=b$, and the given distance of the point C from the vertex A $=c$: put also the abscissa A E $=x$, the ordinate E Q $=y$, and C Q $=s$ *(as before)*.

The area of the triangle ECQ being $(=\frac{1}{2}\text{EC}\times\text{EQ})$ $=\frac{cy-xy}{2}$, its fluxion is therefore $=\frac{c\dot{y}-x\dot{y}-y\dot{x}}{2}$; which added to $y\dot{x}$, the fluxion of the area A E Q, gives $\frac{c\dot{y}+y\dot{x}-x\dot{y}}{2}$ for the fluxion of the whole area A C Q described about the center of force. Whence (by Art. 228) the required centripetal force at Q will be as $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y}+y\dot{x}-x\dot{y}}\big|^{3}}$. Which expression is general, let the curve be of what kind it will. But in the case above, $y$ being $=\frac{b}{a}\sqrt{2ax\pm x^2}$, we have $\dot{y}=\frac{b\dot{x}\times(a\pm x)}{a\sqrt{2ax\pm x^2}}$, $\ddot{y}=\frac{-ab\dot{x}^2}{\overline{2ax\pm x^2}\big|^{\frac{3}{2}}}$, and $c\dot{y}+y\dot{x}-x\dot{y}=$

$\frac{b\dot{x} \times \overline{ca + ax \pm cx}}{a\sqrt{2ax \pm x^2}}$; and therefore, by substituting these values, we get $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3} = \frac{a^4 s}{b^2 \times \overline{ca + ax \pm cx}|^3}$. Which, because $\frac{a^4}{b^2}$ is constant, will also be as $\frac{s}{\overline{ca + ax \pm cx}|^3}$. From whence it follows,

1°. If $c$ be $= \mp a$, or the center of force be in the center of the section, the force itself will be barely as $(\mp s)$ the distance.

2°. If it be in the focus, then $ac + ax \pm cx$ becoming $= CQ \times a$, the force will be inversely as the square of the distance.

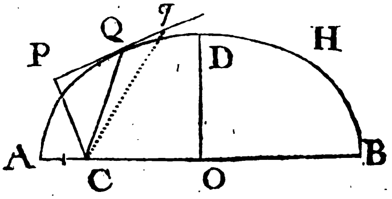
3°. If the given point be in the vertex A, the force will be as $\frac{s}{x^3}$: which therefore in the circle (where $x = \frac{s^2}{2a}$) will be as $\frac{1}{s^5}$, or the fifth power of the distance reciprocally.

4°. Lastly, if the point C be at an indefinite distance from the vertex, or the force be supposed to act in the direction of lines parallel to the axis A O; then the force will be as the cube of O E inversely.

## PROPOSITION X.

236. *To determine the Ratio of the Velocities of Bodies revolving in different Orbits, about the same, or different Centers; the Orbits themselves, and the Forces whereby they are described, being given.*

Let A Q H be any orbit, described about the center of force C, and let the force itself at the principal vertex A be denoted by $F$; also let $r$ stand for the semiparameter, or the ray of curvature at the vertex, and let C P be perpendicular to the tangent Q P.



Then, the celerity at A being always as $\sqrt{rF}$ (by Art. 212) we have $CP : CA :: \sqrt{rF}$ (the velocity at A) to $\frac{CA \times \sqrt{rF}}{CP}$, the velocity at Q (by Art. 225). Which answers in all cases, let the values of A C, $r$ and $F$ be what they will. *Q. E. I.*

### COROLLARY I.

237. If the centripetal force be as the square of the distance inversely, or $F$ be expounded by $\frac{1}{AC^2}$, the velocity at Q will become $\frac{AC}{CP} \times \sqrt{\frac{r}{AC^2}}$, or $\frac{\sqrt{r}}{CP}$: whence *the velocities, in different orbits, about the same center, are in the sub-duplicate ratio of the parameters, and the inverse ratio of the perpendiculars from the center of force to the tangents, conjunctly.*

### COROLLARY II.

238. Hence, if the celerity at Q be denoted by Q $q$, and C $q$ be drawn, then Q $q$ being as $\frac{\sqrt{r}}{CP}$, it follows that $\sqrt{r}$ is as $CP \times Qq$, or as the triangle Q C $q$:

*therefore the areas described about a common center of force in a given time, are in the sub-duplicate ratio of the parameters.*

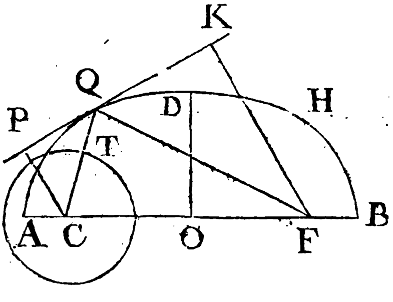
### COROLLARY III.

239. Lastly, since the area of the curve AQHB, &c. when an ellipse,* is known to be as $(AO \times OD)\ AO \times \sqrt{r \times AO}$ (supposing O to be the center) if the same be applied to $\sqrt{r}$, expressing the area described in a given part of time (by the last *Corol.*) we shall thence have $AO \times \sqrt{AO}$, or $AO^{\frac{3}{2}}$ for the measure of the time of one whole revolution. From whence it appears, that *the periodic times, let the species of the ellipsis be what they will, are in the sesquiplicate ratio of their principal axes.*

* Art. 234.

## PROPOSITION XI.

240. *The centripetal Force, tending to a given Point C, being as the Square of the Distances reciprocally, and the Direction and Velocity of a Body at any Point Q, being given; to determine the Path in which the Body moves, and the periodic Time, in case it returns.*



It is evident from Art. 234 and 235, that the trajectory AQB is a conic section; whereof the point C is one of the *foci.*

Let F be the other focus, and upon the tangent P Q K let fall the perpendiculars C P and F K, and let C Q and F Q be drawn: also put the semi-transverse axis $AO = a$, the given focal distance $CQ = d$, and the sine of the angle of direction C Q P (to the radius 1) $= m$; and let the given velocity at Q be to that whereby the body might revolve in a circle about the center C, at that distance, in any given ratio of $n$ to unity: then it will be $n : 1 :: FQ^{\frac{1}{2}} : AO^{\frac{1}{2}}$ (by Art. 234) therefore $n^2 : 1^2 :: FQ\ (2a-d) : AO\ (a)$; whence $AO\ (a)$ is given $= \frac{d}{2-n^2}$. Moreover, since $CP = m \times CQ$, and $FK = m \times FQ$, we have $OD^2$ ($= CP \times FK$) $= m^2 \times CQ \times FQ = \frac{m^2 n^2 d^2}{2-n^2}$; whence the semi-conjugate axis (OD) is given likewise.

Lastly, it will be (by Art. 239) as $CT^{\frac{3}{2}} : AO^{\frac{3}{2}} ::$ $(P)$ the periodic time in any given circle, whose radius is C T, to $\frac{AO^{\frac{3}{2}}}{CT^{\frac{3}{2}}} \times P$ the required time of one revolution when the orbit is an ellipsis; that is, when $n^2$ is less than 2: for, if $n^2$ be $= 2$, the curve (as its axis $\frac{2d}{2-n^2}$ becomes infinite) will degenerate to a parabola; and, if $n^2$ be greater than 2, the axis being negative, it is then an hyperbola; whose two principal diameters are equal to $\frac{2d}{n^2-2}$ and $\frac{2mnd}{\sqrt{n^2-2}}$. *Q. E. I.*

## COROLLARY.

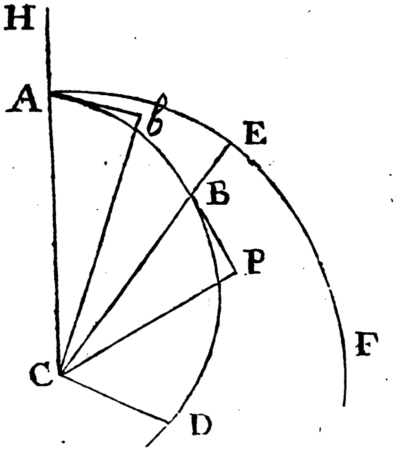
241. Seeing neither the value of A O, nor that of the periodic time, is affected with $m$, it follows that the principal axis, and the periodic time, will remain inva-

riable, if the velocity at Q be the same, let the direction at that point be what it will.

The same solution may likewise be brought out, from Art. 238, by first finding the *principal parameter*: for, it is evident that the area described by the body about the center C, in any given time, is to the area described in the same time, by another body revolving in a circle at the distance CQ, as $mn$ to unity: hence, it will be $1^2 : m^2n^2 :: d : (m^2n^2d)$ the semi-parameter:* from which (proceeding as above) we get $a \times m^2n^2d$ $(= OD^2) = m^2 \times (2ad - d^2)$; and consequently $a = \frac{d}{2-n^2}$, the same as before.

## PROPOSITION XII.

242. *The centripetal Force being as any Power (n) of the Distance, and the Direction and Velocity of a Body at any Point A being given, to determine the Orbit or Trajectory.*



From the center of force C, to any point B in the required trajectory ABD, let CB be drawn; join CA, and let A$b$ be the given direction of the body at the point A, and C$b$ perpendicular thereto; also let the velocity at A be to that whereby a body might describe a circle AEF, about the center C, in any given ratio of $p$ to unity; putting CA$=a$, and CB$=x$: then, because this last velocity (the centripetal force being as $x^n$ or $a^n$) is rightly defined by $a^{\frac{n+1}{2}}$,* the velocity of the body at A will be truly expressed by $pa^{\frac{n+1}{2}}$.

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* Art. 238.

Moreover, it is proved in Art. 221 and 206, that if the celerity, at any given distance $a$ from the center, be denoted by $c$, the celerity at any other distance $x$ will be truly represented by $\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$ : whence, $pa^{\frac{n+1}{2}}$ being substituted for $c$, we have $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}$ for the celerity at B.

But now, to determine the curve itself from hence, let B P be a tangent to it at B, and C P perpendicular to B P; also let C B, produced, meet the periphery of the circle in E; putting the arch A E $= z$, the foresaid velocity at B (to shorten the operation) $= v$, and C$b = b$: then it will be (by Art. 225) $v : c$ (the velocity at A) $:: b : \mathrm{CP} = \frac{bc}{v}$: whence B P ($= \sqrt{\mathrm{CB}^2 - \mathrm{CP}^2}$) $= \frac{\sqrt{x^2v^2 - b^2c^2}}{v}$.

Moreover (by Art. 35) we have, as C B : C P $:: v : \left(\frac{\mathrm{CP}}{\mathrm{CB}} \times v\right)$ the velocity of the body at B in a direction perpendicular to C E; and consequently as C B : C E $:: \frac{\mathrm{CP}}{\mathrm{CB}} \times v$ (the said velocity) to $\frac{\mathrm{CP} \times \mathrm{CE}}{\mathrm{CB}^2} \times v$ the angular velocity of the point E (revolving with the body). By the same article, the velocity at B in the

* Art. 214.

direction C B E will be $\frac{BP}{CB} \times v$: therefore, the velocity of E being to the velocity of B, in the said direction, as $\frac{CP \times CE}{CB^2}$ to $\frac{BP}{CB}$, the fluxions of A E ($\dot{z}$) and CB ($x$) must consequently be in that ratio; that is, $\frac{CP \times CE}{CB^2} : \frac{BP}{CB} :: \dot{z} : \dot{x}$; whence $\dot{z} = \frac{CP \times CE}{CB \times BP} \times \dot{x} = \frac{bc}{v} \times \frac{a}{x} \times \frac{v\dot{x}}{\sqrt{x^2v^2 - b^2c^2}} = \frac{abc\dot{x}}{x\sqrt{x^2v^2 - b^2c^2}} = \frac{ab\dot{x}}{x\sqrt{\frac{x^2v^2}{c^2} - b^2}}$. Which equation is general, let the law of the centripetal force be what it will: but in the case above proposed, $v^2$ being $= \overline{p^2 + \frac{2}{n+1} \times a^{n+1}} - \frac{2x^{n+1}}{n+1}$, and $c^2 = p^2a^{n+1}$; it becomes $\dot{z} = \frac{abp\dot{x}}{x\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2b^2 - \frac{2x^{n+3}}{\overline{n+1} \times a^{n+1}}}}$; whose fluent is the measure of the angular motion; from which, when found, the orbit may be constructed: because, when A E, or the angle A C E is given, as well as C B, the position of the point B is also given. But this value of $\dot{z}$ is indeed too complex to admit of a fluent in algebraic terms, or even by circular arcs and logarithms, except in certain particular cases; as when the exponent $n$ is equal to 1, − 2, − 3, or − 5; besides some others wherein the values of $p$ and $n$ are related in a particular manner. *Q. E. I.*

## COROLLARY I.

243. If the given velocity at A be such that $p^2 + \frac{2}{n+1} = 0$, or $p = \sqrt{\frac{-2}{n+1}}$ (which is always possible when the value of $n+1$ is negative) our equation will become $\dot{z} = \frac{abp\dot{x}}{x\sqrt{-p^2b^2 + \frac{p^2x^{n+3}}{a^{n+1}}}}$: which, by putting $n+3 = m$, &c. is reduced to $\dot{z} = \frac{ab\dot{x}}{x\sqrt{-b^2 + \frac{x^m}{a^{m-2}}}}$: whereof the fluent will be found (by the second part of this work) equal to $\pm \frac{2a}{m}$ multiplied by the difference of the two circular arcs, whose secants are $\frac{x^{\frac{1}{2}m}}{ba^{\frac{1}{2}m-1}}$ and $\frac{a}{b}$, to the radius unity. From this value of the arch A E the position of the point B, in the orbit, is given.

But if the angle of direction C A $b$ be a right one, the fluent will become barely $= \pm \frac{2a}{m}$ × arch whose secant is $\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}m}}$ (because then $b = a$, and the arch whose secant is $\frac{a}{b}$, $= 0$) which therefore when $x^{\frac{m}{2}}$ becomes infinite, will be truly defined by $\pm\frac{1}{2m}\times$ whole periphery A F, &c. Whence it is evident that the body must either fly entirely off, or fall to the center C, in a number of revolutions expressed by $\pm\frac{1}{2m}$; according as the value of $m$ is positive or negative.

Thus, if $n=-2$, and $m=1$, the body will fly entirely off in half a revolution: and, if $n=-4$, and $m=-1$, it will fall to the center in half a revolution.

### COROLLARY II.

244. Moreover, though the fluent expressing the angle at the center cannot be exhibited in a general manner, yet there are certain cases of the exponent ($n$) where its respective values may be derived from each other.

For let (as above) $n+3$ be put $=m$, and (to shorten the operation) let C A ($a$) be taken as unity: then our equation will be transformed to $\dot{z}=\frac{b\dot{x}}{x\sqrt{\overline{1+\frac{2}{\overline{m-2}.p^2}}\times x^2-b^2-\frac{2x^m}{\overline{m-2}.p^2}}}$: make $y=x^{\frac{m}{2}}$, and it will be farther transformed to $\dot{z}=\frac{2}{m}\times\frac{b\dot{y}}{y\sqrt{\overline{1+\frac{2}{\overline{m-2}.p^2}}\times y^{\frac{4}{m}}-b^2-\frac{2y^2}{\overline{m-2}.p^2}}}$: put $r=\frac{4}{m}$, and it will become $\dot{z}=\frac{2}{m}\times\frac{b\dot{y}}{y\sqrt{\frac{ry^2}{\overline{r-2}.p^2}-b^2+\overline{1-\frac{r}{\overline{r-2}.p^2}}\times y^r}}$: lastly, let $\frac{r}{\overline{r-2}.p^2} = 1 + \frac{2}{\overline{r-2}.q^2}$ (or $1 - \frac{r}{\overline{r-2}.p^2} = -\frac{2}{\overline{r-2}.q^2}$, or $q^2 = \frac{2p^2}{r-p^2 \times \overline{r-2}}$) and then we shall have $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{1 + \frac{2}{\overline{r-2}.q^2} \times y^2 - b^2 - \frac{2y^r}{\overline{r-2}.q^2}}}$. Which expression (excepting the general multiplicator $\frac{2}{m}$) being exactly of the same form with the first above given must, therefore be the fluxion of the angle at the center, when the index of the force is $r-3$; for the very same reasons that the former appears to be the fluxion thereof when the index is $m-3$ (or $n$).

Hence, if the fluent of

$$\frac{b\dot{y}}{y\sqrt{1 + \frac{2}{\overline{r-2}.q^2} \times y^2 - b^2 - \frac{2y^r}{\overline{r-2}.q^2}}},$$

or the angle at the center, when the exponent is $r-3$ (or $\frac{4}{m} - 3 = \frac{4}{n+3} - 3$) be denoted by $w$, the value of $z$, (the measure of the said angle, when the exponent is $m-3$ or $n$) will be truly defined by $\frac{2w}{m}$.

From which we collect, that if the indices of the force, in any two cases, be represented by $n$ and $\frac{4}{n+3} - 3$, and the respective distances from the center by $x$ and $x^{\frac{n+3}{2}}$, then the angles themselves corresponding to those distances will be every where in the constant ratio of $2$ to $n+3$. Therefore, when the orbit can be constructed in the one case, it also may in the other, provided the above equation $q^2$ $\left(=\frac{2p^2}{\overline{r-p^2}\times\overline{r-2}}\right)=\frac{\overline{n+3}.p^2}{2+\overline{n+1}.p^2}$, for the relation of the celerities at A, does not become impossible, as it will, sometimes, when $n$ is a negative number.

## COROLLARY III.

245. If the body be supposed to move in a vertical direction A H; then, putting the velocity $\sqrt{\overline{p^2+\frac{2}{n+1}}\times a^{n+1}-\frac{2x^{n+1}}{n+1}}=0$, we get $x$ (C H) $=\overline{\tfrac{1}{2}p^2\times\overline{n+1}+1}\Big|^{\frac{1}{n+1}}\times a=$ the height to which the body will ascend: hence $\overline{\tfrac{1}{2}p^2\times\overline{n+1}+1}\Big|^{\frac{1}{n+1}}\times a-a$ (= AH) is the distance through which it must freely descend to acquire the given celerity at A: this distance, in case of an uniform force, when $n=0$, will become $=\tfrac{1}{2}p^2a$: and, when the force is inversely as the square of the distance, it will then be $=\frac{p^2a}{2-p^2}$.

But, when $p=1$, or the velocity at A is just sufficient to retain a body in the circle AEF, AH becomes $=\overline{\frac{3+n}{2}}\Big|^{\frac{1}{n+1}}\times a-a$: which in the two cases aforesaid will be equal to $\tfrac{1}{2}a$, and $a$ respectively; but, infinite, when $n$ is $=-3$.

## COROLLARY IV.

246. When the value of $n+1$ is positive, the velocity at the center, where $x=0$, will be barely $= \sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; but if the value of $n+1$ be negative, the velocity at the center will be infinite; because, then $0^{n+1}$ is infinite.

## COROLLARY V.

247. Moreover, when $n+1$ is negative and $x$ infinite, the velocity also becomes $= \sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; because then $x^{n+1}=0$.

Hence, if the centripetal force be inversely as some power of the distance greater than the first, the body may ascend, *ad infinitum*, and have a velocity always greater than $\sqrt{p^2+\frac{2}{n+1}\times a^{n+1}}$; which is to $pa^{\frac{n+1}{2}}$, the given velocity, at A, as $\sqrt{p^2+\frac{2}{n+1}}$ to $p$. And this will actually be the case when the value of $p^2+\frac{2}{n+1}$ is positive, or $p^2$ greater than $\frac{2}{-n-1}$, but not otherwise, the square root of a negative quantity being impossible.

Thus, if $n=-2$, or the force be inversely as the square of the distance, and $p^2$, at the same time, greater than $2\left(\frac{2}{-n-1}\right)$ the body will not only contimue to ascend *in infinitum*, but have a velocity always greater than that defined by $\sqrt{p^2-2}$, which is its limit.

### COROLLARY VI.

248. Hence the least celerity sufficient to cause the body to ascend for ever in a right-line is given. For, putting $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}} = 0$, we have $p = \sqrt{\frac{2}{-n-1}}$. Therefore the least celerity by which the body might ascend for ever, is to that whereby it may revolve in a circle A E F, as $\sqrt{\frac{2}{-n-1}}$ to unity. From which it appears that, *if the force be inversely as any power of the distance greater than the third, a less velocity will cause a body to ascend* ad infinitum *than would retain it in a circle.*

### SCHOLIUM.

249. From the ratio of the velocity $\left(\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}\right)$ wherewith the body arrives at any distance $x$ from the center, to that $\left(x^{\frac{n+1}{2}}\right)$* which it ought to have to revolve in a circle at the same distance, it will not be difficult to determine in what cases the body will be forced to the center, and in what others it will continue to fly it *ad infinitum.*

For, first, if the angle C A $b$ be acute, or the body from A begins to descend, it will continue to do so till it actually arrives at the center, if the former velocity, during the descent, be not somewhere greater than the latter, or the quotient $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ greater than unity; because, if it ever begins to ascend, it must have an *apse*, as D (where a right-line drawn from the center cuts the orbit at right-angles) and there the celerity must evidently be greater than that sufficient to cause the body to revolve in a circle.

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* Art. 214.

Secondly, but if the quantity $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$, in the access of the body towards the center, increases so as to become greater than unity, or be every where so; then the velocity at all inferior distances being more than sufficient to retain a body in a circle at any such distance, the projectile cannot be forced to the center.

After the same manner, if the angle C A $b$ be obtuse, or the body from A begins to ascend, it will continue to do so for ever, when the foresaid quantity is always greater than unity, or, which is the same, when the body, in its recess from the center, has in every place through which it passeth, a velocity greater than sufficient to retain it in a circle at that distance.

It therefore now remains to find in what laws of the centripetal force these different cases obtain: and, first, it is easy to perceive that when the value of $n+1$ is positive, that of $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ will, by increasing $x$, become equal to nothing. Therefore the body cannot ascend for ever in this case: neither can it descend to the center (except in a right-line) because the foresaid quantity, by diminishing $x$, becomes greater than unity (or any other assignable magnitude).

But, if the value of $n$ be betwixt $-1$, and $-3$, the said general expression, taking $x$ infinite, will also become infinite, provided the value of $p^2 + \frac{2}{n+1}$ be positive (or $p^2$ greater than $\frac{2}{-n-1}$). Therefore the body, in this case, may ascend *ad infinitum*, but cannot possibly fall to the center (except in a right-line) since, $\sqrt{-\frac{2}{n+1}}$, the value of the general expression, when $x=0$, is greater than unity.

Lastly, if $n$ be expressed by any negative number greater than $-3$, or the law of the force be inversely as any power of the distance greater than the third, the two extreme values of $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ will, *still*, be denoted as in the preceding case; but here the latter of them, $\sqrt{\frac{-2}{n+1}}$, is less than unity. Therefore the body must, in this case, either ascend for ever, or be forced to the center; except in one particular circumstance, hereafter to be taken notice of.

Now, from these observations we gather,

1°. That, when the centripetal force is as any power of the distance directly, or less than the first power thereof inversely, the orbit will always have an higher and a lower *apse*; beyond which the body cannot ascend or descend.

2°. That, when the centripetal force is inversely as any power of the distance (whole or broken) betwixt the first and third, the orbit will also have two *apsides*, if $p$ be less than $\sqrt{-\frac{2}{n+1}}$; but otherwise, only one; in which last case, the body, after it has passed its apse, will continue to recede from the center *in infinitum*.

3°. That when the force is inversely as any power greater than the third, the orbit can, at most, have but one *apse*; but, in some cases, it will have none at all; and it may be worth while to inquire here, under what restrictions of the velocity *(p)* this will happen; since thereby, besides being able to know when the body will be forced to the center, &c. we shall fall upon a circumstance somewhat remarkable and curious.

Now it appears, that, if the body from A begins to descend, it must, when it comes to an *apse* at D, have a velocity there greater than is sufficient to retain it in a circle; in which case the general expression $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ (so often mentioned above) must accordingly be greater than unity. Let it be therefore made equal to unity, which is the utmost limit thereof, beyond which the orbit cannot admit of an *apse;* putting at the same time $\dot{x}$, or its divisor $\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2b^2 - \frac{2x^{n+3}}{\overline{n+1}\,.\,a^{n+1}}}$, in the general equation of the orbit, equal to nothing (it being always so at the *apsides).* Then, from these two equations, duly ordered, we shall get $x = \overline{\frac{2 + \overline{n+1}\,.\,p^2}{n+3}}\Big|^{\frac{1}{n+1}} \times a$, and $p^2 \left(= \frac{x^{n+3}}{a^{n+1}}\right) = \overline{\frac{2 + \overline{n+1}\,.\,p^2}{n+3}}\Big|^{\frac{n+3}{n+1}} \times \frac{a^2}{b^2}$. Now, it is evident, if the value of $p$ be greater than is given from the last equation, the orbit will have an apse; but, if less, it can have none. In the former case, the body will therefore fly quite off; and in the latter, it will be forced to the center. But we are now, naturally, led to inquire what will be the consequence when the value of $p$ is neither greater nor less, but exactly the same as given from the foresaid equation: this is the case above hinted at; and here the body will continue to descend for ever in a spiral, yet never so low as to enter within the circle whose radius C D is $= \overline{\frac{2 + \overline{n+1}\,.\,p^2}{n+3}}\Big|^{\frac{1}{n+1}} \times a$. For, if the contrary were possible, the body, at its arrival to the circumference of that circle, would (because of the foresaid equations) not only have a direction, but also velocity proper to retain it therein; which cannot be, because the parts of the orbit on either side of an apse are always similar to each other.

From the same equation, the value of the limit will also be given when the angle of direction C A $b$ is obtuse, or the body is projected upwards.

For that equation (as is easy to demonstrate)* admits of two different roots, or values of $p$; the one greater, the other less, than unity: whereof the former, giving C D $(x)$ less than C A, is to be taken in the preceding case, and the latter (making C D greater than C A) in the present. And the body will, either, continue to ascend for ever, or come to an *apse*, and from thence fall to the center, according as the given value of $p$ is greater or less than that here specified. But if it be neither greater nor less, but exactly the same, then the body, though it will still continue to ascend for ever in a spiral, yet it can never rise so high as the circumference of the circle whose radius C D is =

$$\overline{\frac{2+\overline{n+1}\,.\,p^2}{n+3}}\Bigg|^{\frac{1}{n+1}} \times a,$$ for reasons similar to those already delivered, in respect to the preceding case.

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[*] Mathematical Dissertations, p. 167.

# APPENDIX

# TO

# *VOLUME THE FIRST.*

IN presenting the student with an Appendix, the object of the editor is to furnish him with such additional matter as the progress in this subject, and others dependent upon it, made since the original publication of the work, seems to demand. Many important discoveries in the Fluxionary Calculus, under different titles, have since that period been published, both by our author himself and contemporaries, and subsequent writers. The applications of these improvements to questions connected with Natural Philosophy, have been equally extensive, and in order to understand the great works of Lagrange and Laplace, and many other important works, which are treated in a manner wholly analytical, it has become necessary to have at command every resource which this branch of abstract inquiry, in its present improved state, can afford. As far, therefore, as the limits of an elementary work will permit, we will endeavour to supply these *desiderata*, making, at the same time, such remarks upon the work itself as may appear useful.

SECT. I.—In this Section we have the fluxions of algebraic quantities only. The fluxions of logarithms, of exponentials, and of circular quantities, are given in articles 126, 250, and 142, respectively. In the examples we intend to give in *maxima* and *minima*, in *drawing tangents to curves*, &c. &c. (subjects which are introduced previously to those articles) it being necessary to employ the fluxions of such quantities, we shall proceed to determine them.

### 1. *Required the Fluxion of $a^x$, a being constant, and x variable.*

By ordinary algebra

$$a^x = 1 + Ax + \frac{A^2x^2}{1\,.\,2} + \frac{A^3x^3}{1\,.\,2\,.\,3} + \ldots\ldots.\infty \text{ where}$$

$A = a - 1 - \frac{1}{2}\,.\,(a-1)^2 + \frac{1}{3}\,.\,(a-1)^3 - \ldots\ldots\infty$ .*

Therefore, taking the fluxions of each term

$$\begin{aligned}(a^x)^{\cdot} &= A\dot{x} + A^2.\,x\dot{x} + \frac{A^3x^2\dot{x}}{1.\,2} + \frac{A^4.\,x^3\dot{x}}{1.\,2.\,3} + \ldots\ldots. \\ &= A\dot{x} \times (1 + Ax + \frac{A^2x^2}{1.\,2} + \ldots\ldots) \\ &= A\dot{x} \times a^x.\end{aligned}$$

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* For $a^x = (1 + \overline{a-1})^x = 1 + x\,.\,(a-1) + x\,.\,\frac{x-1}{2} \times$
$(a-1)^2 + \ldots\ldots.$
$= 1 + x\,\{(a-1) - \frac{1}{2}.\,(a-1)^2 + \frac{1}{3}\,.\,(a-1)^3$
$- \ldots\ldots\} + Bx^2 + Cx^3 +$ &c.

*B*, *C*, &c. being at present unknown.

$\therefore a^x = 1 + Ax + Bx^2 + Cx^3 + \ldots\ldots$

Similarly $a^{x+u} = 1 + A.(x+u) + B.(x+u)^2 + C.(x+u)^3$
Also $a^{x+u} = a^x \times a^u = (1 + Ax + Bx^2 + \ldots) \times (1 + Au + Bu^2 + \ldots)$
$= 1 + A\,.\,(x+u) + A^2\,.\,xu + B\,.\,(x^2 + u^2)$
$+ AB(x^2a + u^2x) + \ldots\ldots..$ by actual multiplication.

Hence $A^2.\,xu + BA\,(x^2u + u^2x) + \ldots\ldots = 2\,Bxu +$
$3\,C\,(x^2u + u^2x) +$ &c.

$\therefore B = \frac{A^2}{1.\,2}, C = \frac{BA}{3} = \frac{A^3}{1.\,2.\,3}$, and in the same manner may the other indeterminates be found.

Now $A=a-1-\frac{1}{2}.(a-1)^2+.....\infty = l.(a)$ ($l$ being the characteristic of hyperbolic logarithms).

$\therefore$ *the fluxion of an exponential* $(a^x)$ *is equal to the product of the exponential itself, the hyperbolic logarithm of the constant, and the fluxion of the variable.*

Let $a = e =$ the hyperbolic base. Then $le = 1$ and $(e^x)^{\cdot} = e^x\dot{x}$.

### (2.) *To determine the Fluxion of log. x, a being the base of the System.*

Let $u=log.\ x$. Then $x=a^u$,
and $\dot{x}=(a^u)^{\cdot}=la\times\dot{u}\times a^u=xla\times\dot{u}$ (by 1)

$$\therefore\ \dot{u}=\frac{\dot{x}}{x\ la}.$$

Or, *The fluxion of a logarithm equals the fluxion of the variable, divided by the product of the variable and the hyperbolic logarithm of the base of the system.*

If $l.a = 1$, or $a =$ the hyperbolic base, we have $\dot{u} = (lx)^{\cdot} = \frac{\dot{x}}{x}$.

### (3.) *To find the fluxion of $x^v$.*

Let $u=x^v$
Then $l.u=v.l.x$. Therefore

$$\frac{\dot{u}}{u}=\dot{v}\,l.x + (lx)^{\cdot}v=\dot{v}lx+\frac{\dot{x}v}{x}\ (2)$$

$$\therefore\ \dot{u}=\dot{v}x^v lx+\dot{x}vx^{v-1}.$$

### (4.) *To find the fluxion of sin. x, and of cos. x.*

By Demoivre's Theorem

$$\left.\begin{aligned}cos.\ n\theta+\sqrt{-1}.\ sin.\ n\theta&=(cos.\ \theta+\sqrt{-1}.\ sin.\ \theta)^n\\ cos.\ n\theta-\sqrt{-1}\ sin.\ n\theta&=(cos.\ \theta-\sqrt{-1}\ sin.\ \theta)^n\end{aligned}\right\}*$$

---

* This Theorem is best proved by the actual multi-

Adding and subtracting these equations, after expanding their right-hand members by the binomial theorem, and dividing by 2, and $2\sqrt{-1}$ respectively, we get

$$\cos. n\theta = \cos.^{n}\theta - \frac{n.(n-1)}{2}\cos.^{n-2}\theta\ \sin.^{2}\theta + \frac{n.(n-1)\ (n-2)\ (n-3)}{2\ .\ 3\ .\ 4}\cos.^{n-4}\theta\ .\sin.^{4}\theta - \&c.$$

and $\sin. n\theta = n\ .\cos.^{n-1}\theta\ \sin.\ \theta - \frac{n.\ (n-1)\ (n-2)}{2\ .\ 3}$

$$\times \cos.^{n-3}\theta\ \sin.^{3}\theta + \frac{n.(n-1)\,(n-2)\,(n-3)\,(n-4)}{2.\,3.\,4.\,5}$$

$$\times \cos.^{n-5}\theta\ .\sin.^{5}\theta\ \&c.$$

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plication of $\cos.\ A \pm \sqrt{-1}\ \sin.\ A$, $\cos.\ B \pm \sqrt{-1} \times \sin.\ B$, $\cos.\ C \pm \sqrt{-1}\ \sin.\ C$........to $n$ terms. The product will be

$\cos.\ (A+B+......n$ terms$) \pm \sqrt{-1}\ .\ \sin.\ (A+B+C+'......n$ terms$) = (\cos.\ A \pm \sqrt{-1}\ .\ \sin.\ A) \times (\cos.\ B \pm \sqrt{-1}\ \sin.\ B) \times$ &c. Let $A = B = C = ........$

Then $\cos.\ n\,A \pm \sqrt{-1}.\sin.\ n\,A = (\cos.\ A \pm \sqrt{-1}\ .\ \sin.\ A)^{n}$ which is the theorem in the case of $n$ being an integer.

Again, let $a = \frac{p}{q}\ .\ A$.

Then $(\cos.\ a \pm \sqrt{-1}\ \sin.\ a)^{q} = \cos.\ qa \pm \sqrt{-1} \times \sin.\ qa = \cos.\ pA \pm \sqrt{-1}\ \sin.\ pA = (\cos.\ A \pm \sqrt{-1}\ \sin.\ A)^{p}$.

$\therefore \cos.\frac{p}{q}\ A \pm \sqrt{-1}\ \sin.\frac{p}{q}\ A = (\cos.\ A \pm \sqrt{-1}\ \sin\ A)^{\frac{p}{q}}$ which is the other case.

Let $n=\infty$, and $\theta=0$, so that $n\theta$ may be finite and $=x$. Then $cos.\ \theta=1$, $sin.\ \theta=\theta=\frac{x}{n}$, $\frac{n.(n-1)}{2}=\frac{n^2}{2}$ &c. &c. and by substitution we find

$$\left.\begin{array}{l} cos.\ x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\ldots\ldots. \\ sin.\ x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1\ldots\ldots7}+\ldots\ldots. \end{array}\right\}$$

Hence

$$(sin.\ x)^{\cdot}=\dot{x}\,(1-\frac{x^2}{1.2}+\frac{x^3}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\ \ldots\ldots)$$

$$=\dot{x}\ cos.\ x$$

and

$$(cos.\ x)^{\cdot}=-\ \dot{x}\ (x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\ \ldots\ldots)$$

$$=-\ \dot{x}\ sin.\ x.$$

Hence, *The fluxion of the sine of an arc equals the product of the fluxion of the arc and the cosine of the arc.* Also, *The fluxion of the cosine of an arc equals minus the product of the fluxion of the arc and the sine of the arc.*

These results are derived from the circle in a more simple manner by our author in Art. 142.

5. *To find the fluxion of tan. x, cot. x, sec. x, and cosec. x.*

$$(tan.\ x)^{\cdot}=\left(\frac{sin.\ x}{cos.\ x}\right)^{\cdot}=\frac{(sin.\ x)^{\cdot}}{cos.\ x}-\frac{(cos.\ x)^{\cdot}\ sin.\ x}{cos.\ ^2x}$$

$$=\frac{\dot{x}\ cos.\ x}{cos.\ x}+\frac{\dot{x}\ sin.\ ^2x}{cos.\ ^2x}=\frac{\dot{x}.\ (cos.\ ^2x+sin.\ ^2x)}{cos.\ ^2x}$$

$$=\frac{\dot{x}}{cos.\ ^2x}\ \text{(radius = 1), or }=\dot{x}\,.\ sec.\ ^2x\ \text{ or }=$$

$$\dot{x}\,.\ (1+tan.\ ^2x).$$

Again $(cot.\ x)^{\cdot} = (tan.\ \overline{90^\circ - x})^{\cdot} = \frac{-\dot{x}}{cos.\ ^2(90-x)} = \frac{-\dot{x}}{sin.\ ^2x}$, or $= -\dot{x}\,.\,(1+cot.\ ^2x)$.

Also $(sec.\ x)^{\cdot} = \left(\frac{1}{cos.\ x}\right)^{\cdot} = \frac{-(cos.\ x)^{\cdot}}{cos.\ ^2x} = \frac{\dot{x}\ sin.\ x}{cos.\ ^2x}$, or $= \dot{x}\ tan.\ x\,.\,sec.\ x$, or $= \dot{x}\sqrt{sec.\ ^2x-1}\,.\,sec.\ x$.

And $(cosec.\ x)^{\cdot} = \{sec.\ (90^\circ - x)\}^{\cdot} = -\dot{x}\,.\,\frac{sin.\ (90-x)}{cos.\ ^2(90-x)} = \frac{-\dot{x}\,.\,cos.\ x}{sin.\ ^2x}$, or $= -\dot{x}\,.\,cot.\ x\,.\,cosec.\ x$, or $= -\dot{x}\,.\,\sqrt{cosec.\ ^2x-1}\,.\,cosec.\ x$.

These results, as well as those deduced in the preceding articles, being of frequent occurrence, the student is recommended to commit them to memory. They will enable him to find the fluxions of the most complicated forms, such as $l\,.\,\frac{e^x-1}{e^x+1}$, $l.\sqrt{\frac{1+sin.\ x}{1-sin.\ x}}$, $\frac{x}{2} - \frac{sin.\ x\,.\,cos.\ x}{2}$, $\frac{sin.\ nx}{(sin\ x)^n}$, $cos.\ \left(l\,.\,\frac{1}{x}\right)$ &c., whose fluxions, after the proper reductions, are found to be

$$\frac{2e^x\dot{x}}{e^{2x}-1},\ \frac{\dot{x}}{cos.\ x},\ \dot{x}\ sin.\ ^2x,\ \frac{-n.\ sin.\ (n-1)\,x\,.\,\dot{x}}{(sin.\ x)^{n+1}},$$

$\frac{\dot{x}}{x}\,sin.\ \left(l\,.\,\frac{1}{x}\right)$, &c. respectively.

Having prepared the student for finding the fluxions of every species of quantity, we will show him the utility of some very celebrated theorems.

*Let $f(x)$ represent a quantity any how involving $x$ (called a function of $x$), and let it be required to expand, when it is possible, $f(x)$ in terms of the ascending powers of $x$.*

Assume $u = fx = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots\ldots$ $A$, $B$, $C$, &c. being constant.

Then $\frac{\dot{u}}{\dot{x}} = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots\ldots$

$$\frac{\ddot{u}}{\dot{x}^2} = 2C + 3.2Dx + 4.3Ex^2 + 5.4Fx^3 + \ldots\ldots$$

$$\frac{\dddot{u}}{\dot{x}^3} = 3.2D + 4.3.2Ex + 5.4.3Fx^2 + \ldots\ldots$$

&c. = &c. the law being evident.

Let $x = 0$, and the corresponding values of $u$, $\frac{\dot{u}}{\dot{x}}$, $\frac{\ddot{u}}{\dot{x}^2}$, &c. be denoted by $U$, $U_1$, $U_2$, &c.

Then $A = U$, $B = U_1$, $C = U_2 \times \frac{1}{2}$, $D = U_3 \times \frac{1}{2.3}$, &c. = &c.

Therefore $f(x) = U + U_1 . x + U_2 . \frac{x^2}{2} + U_3 . \frac{x^3}{2.3}$ + &c. which is the Theorem of Maclaurin.

Ex. 1. *Required to expand* $l.(1+x)$.

$$u = l.(1+x), \therefore U = l.(1) = 0$$

$$\frac{\dot{u}}{\dot{x}} = \frac{1}{1+x}, \therefore U_1 = \frac{1}{1} = 1$$

$$\frac{\ddot{u}}{\dot{x}^2} = -\frac{1}{(1+x)^2}, \therefore U_2 = -1$$

$$\frac{\dddot{u}}{\dot{x}^3} = \frac{2}{(1+x)^3}, \therefore U_3 = 2$$

$$\frac{\ddddot{u}}{\dot{x}^4} = \frac{-2.3}{(1+x)^4}, \therefore U_4 = -2.3$$

&c. = &c., &c = &c.

Therefore $l.(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots..$

Ex. 2. *Required to express an arc in terms of its sine.*

Let $x = \sin. u$

Then $u =$ arc whose sine is $x$, $\therefore U = 0$

$$\frac{\dot{u}}{\dot{x}} = \frac{1}{\cos. u} = \frac{1}{\sqrt{1-x^2}} \text{ (see 4), } \therefore U = 1 \text{ (radius being 1)}$$

$$\frac{\ddot{u}}{\dot{x}^2} = \frac{x}{(1-x^2)^{\frac{3}{2}}}, \therefore U_2 = 0$$

$$\frac{\dddot{u}}{\dot{x}^3} = \frac{1}{(1-x^2)^{\frac{3}{2}}} + \frac{3 . x^2}{(1-x^2)^{\frac{5}{2}}}$$

$$= \frac{3}{(1-x^2)^{\frac{5}{2}}} - \frac{2}{(1-x^2)^{\frac{3}{2}}}, \therefore U_3 = 1$$

$$\frac{\ddddot{u}}{\dot{x}^4} = \frac{x}{(1-x^2)^{\frac{7}{2}}} \times (9 + x^2), \therefore U_4 = 0$$

$$\frac{\overset{\cdot\cdot\cdot\cdot\cdot}{u}}{\dot{x}^5} = 3 . 3 . (1-x^2)^{-\frac{5}{2}} + 2 . 5 . 9 x^2 (1-x^2)^{-\frac{7}{2}} +$$

$$3 . 5 . 7 x^4 (1-x^2)^{-\frac{9}{2}}, \therefore U_5 = 3 . 3$$

&c. = &c.

$$\text{Hence } u = x + \frac{x^3}{1 . 2 . 3} + \frac{3 . 3 x^5}{1 . 2 . 3 . 4 . 5} + \text{\&c.}$$

Ex. 3. *Required to express an arc in terms of its tangent.*

Let $x = \tan. u$.

Then, since $\frac{\dot{u}}{\dot{x}} = (1 + u^2)^{-1}$, (by 5) by proceeding as above, we get

$$u = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots$$

Other examples are, *To express a number in terms of its logarithm; To express* $(\cos. x)^n$ *in terms of* $x$; *To express* $\frac{e^x}{\cos. x}$ *in terms of* $x$; &c. &c.

In many cases this Theorem fails, as in $u = \frac{1}{x^n}$, $u = l(x)$, $x = \sec. u$, $x = \operatorname{cosec}. u$, &c. because in the first $U = \frac{1}{0} = \infty$, in the second $U = l.(0) = -\infty$, in the third and fourth $x$ can never equal 0. In many of these cases a transformation will give the function the form proper for developement. We have not room, however, to produce any instances.

Supposing $u = f(x)$ to be such a function of $x$, as is developable into a series of powers of $x$, with constant co-efficients, we may assume

$$u = f(x) = Ax^a + Bx^b + Cx^c + \ldots\ldots\ldots$$

Let $x$ become, by the variation of $x$, $x + h$. Then

$$\begin{aligned}\acute{u} = f.(x+h) &= A.(x+h)^a + B.(x+h)^b + C.(x+h)^c + \ldots \\ &= Ax^a + Bx^b + Cx^c + \ldots\ldots\ldots \\ &+ h.(aAx^{a-1} + bBx^{b-1} + cCx^{c-1} + \ldots\ldots) \\ &+ \frac{h^2}{1.2}.(a.\overline{a-1}.Ax^{a-2} + b.\overline{b-1}.Bx^{b-2} \\ &\quad + \ldots\ldots) \\ &+ \frac{h^3}{1.2.3}.(a.\overline{a-1}.\overline{a-2}.A.x^{a-3} + b.\overline{b-1} \times \\ &\quad \overline{b-2}.Bx^{b-3} + \ldots\ldots\ldots)\end{aligned}$$

+ &c. &c. by the binomial theorem, which we must suppose established on algebraical principles.

But $Ax^a + Bx^b + \ldots\ldots = f(x) = u$

$$aAx^{a-1} + bBx^{b-1} + \ldots\ldots\ldots = \frac{\dot{u}}{\dot{x}}$$

$$a.\overline{a-1}.\,Ax^{a-2}+b.\overline{b-1}.\,Bx^{b-2}+\ldots\ldots=\frac{\ddot{u}}{\dot{x}^2}$$

$$\&c. = \&c.$$

$$\therefore \acute{u}=f.\,(x+h)=u+\frac{\dot{u}}{\dot{x}}h+\frac{\ddot{u}}{\dot{x}^2}\cdot\frac{h^2}{1.2}+\frac{\dddot{u}}{\dot{x}^3}\cdot\frac{h^3}{1.2.3}$$

+ &c. which was first given by Newton at the end of his Principia, but is known by the name of *Taylor's Theorem.* Lagrange and some subsequent writers make this theorem the basis of the Fluxional or Differential Calculus, by defining the fluxion or differential of a function to be the second term of its developement according to that theorem.

The theorem may be simplified by making $h=\dot{x}$; for then

$$f.\,(x+\dot{x}) = u+\dot{u}+\frac{\ddot{u}}{1.2}+\frac{\dddot{u}}{1.2.3}+\frac{\ddddot{u}}{1.2.3.4}+\&c.$$

Functions of various kinds may be expanded by this theorem, into a variety of series, by giving to *(h)* the requisite forms, &c.

We will give an example.

Let $u = tan.^{-1}x$, (or the arc whose tangent is $x) = \frac{\pi}{2} - y$, by supposition.

Then $x=tan.\,u=tan.\left(\frac{\pi}{2}-y\right)=cot.\,y$

$$\frac{\dot{u}}{\dot{x}}=\frac{1}{1+x^2}=sin.^2y=-\frac{\dot{y}}{\dot{x}}$$

$$\frac{\ddot{u}}{\dot{x}^2}=\frac{\dot{y}}{\dot{x}}\times sin.\,2y=-(sin.\,y)^2.\,sin.\,2y$$

Similarly $\frac{\dddot{u}}{1.2.\dot{x}^3}=-\frac{\dot{y}}{\dot{x}}(sin.\,y\,.\,cos.\,y\,.\,sin.\,2y+$

$sin.^2y\,.\,cos.\,2y)$

$=sin.^3y.\,sin.\,3y$

and $\frac{\ddddot{u}}{1.2.3.x^4} = -\sin.^4y.\sin.4y$

&c. = &c.

Hence $\tan.^{-1}(x+h) = \tan.^{-1}x + \sin.y.\sin.y.\frac{h}{1}$
$-\sin.^2y.\sin.2y.\frac{h^2}{2} + \sin.^3y.\sin.3y.\frac{h^3}{3} - \sin.^4y \times$
$\sin.4y.\frac{h^4}{4} + \dots\dots\dots$ (a)

Let $h = -x$.

Then, $\tan.^{-1}x = \frac{\pi}{2} - y = x.\sin.y.\sin.y + \frac{x^2}{2} \times$
$\sin.2y.\sin.^2y + \frac{x^3}{3}\sin.3y.\sin^3y + \frac{x^4}{4}.\sin.4y \times$
$\sin.^4y + \dots\dots\dots\dots\dots\dots$ (b)

Since $x = \cot.y$, we also have

$$\frac{\pi}{2} = y + \sin.y.\cos.y + \frac{\sin.2y}{2}.\cos.^2y + \dots\dots\dots \text{(c)}$$

Again, in series (a) let $h = -2x$, and we get

$2\tan.^{-1}x = \frac{2x}{1}.\sin.y.\sin.y + \frac{2^2x^2}{2}.\sin.2y \times$
$\sin.^2y + \frac{2^3x^3}{3}\sin.3y.\sin.^3y + \dots\dots\dots\dots\dots\dots$ (d)

Substituting $\cot.y$ for $x$ in (d) we get

$\frac{\pi}{2} = y + \sin.y.\cos.y + \frac{2}{2}.\sin.2y.\cos.^2y + \frac{2^2}{3} \times$
$\sin.3y.\cos.^3y + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$ (e)

Hence, by making $y = \frac{\pi}{4}$, we have

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{1}{2} + \frac{2}{2 \times 2} + \frac{2}{3.2} - \frac{2^2}{5.2} - \frac{2^3}{6.2} - \frac{2^3}{7.2} + \text{\&c.}$$

$$\therefore \frac{\pi}{2} = 1 + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \&c. \ldots\ldots (f.)$$

And by similar substitutions in series *(a)*, many other beautiful formulæ may be obtained. See Euler, Inst. Calc. Diff. Part II, 57, &c.

Maclaurin's might evidently have been deduced from this theorem.

*Given* $y = a + x . \varphi y$, *to expand* $u = fy = f . (a + x\varphi y)$ *according to the powers of* $x$, $\varphi$ *and* $f$ *denoting known functions, and* $a$ *being independent of* $x$ *and* $y$.

Since $y = a + x . \varphi y$, we have

$$\frac{\dot{y}}{\dot{x}} = v + \frac{x\dot{v}}{\dot{x}} \; (\varphi y = v)$$

$$\frac{\ddot{y}}{\dot{x}^2} = \frac{2\dot{v}}{\dot{x}} + x . \frac{\ddot{v}}{\dot{x}^2}$$

$$\frac{\dddot{y}}{\dot{x}^3} = \frac{3\ddot{v}}{\dot{x}^2} + x . \frac{\dddot{v}}{\dot{x}^3}$$

Now, since $v$ is a function of $\dot{y}$, and $\dot{y}$ is evidently a function of $x$, taking the fluxions in the supposition that $x$ is the principal variable, we have

$\frac{\dot{v}}{\dot{x}} = \frac{\dot{v}}{\dot{y}} . \frac{\dot{y}}{\dot{x}}$ (this will be manifested by an example, such as $v = fy = f . (\varphi x) = (a + x)^n$, where $\dot{v} = n \times (a + x)^{n-1} . (a + x)^{\cdot} = n . (a + x)^{n-1}\dot{x}$).

$$\therefore \frac{\ddot{v}}{\dot{x}^2} = \left(\frac{\dot{v}}{\dot{y}}\right)^{\cdot} . \frac{\dot{y}}{\dot{x}^2} + \frac{\dot{v}}{\dot{y}} . \frac{\ddot{y}}{\dot{x}^2}$$

$$= \frac{\ddot{v}}{\dot{y}^2} . \frac{\dot{y}^2}{\dot{x}^2} + \frac{\dot{v}}{\dot{y}} . \frac{\ddot{y}}{\dot{x}^2}$$

$$\&c. = \&c.$$

$$\therefore \frac{\dot{y}}{\dot{x}} = v + x . \frac{\dot{v}}{\dot{y}} . \frac{\dot{y}}{\dot{x}}$$

$$\frac{\ddot{y}}{\dot{x}^2} = 2.\frac{\dot{v}}{\dot{y}}\cdot\frac{\dot{y}}{\dot{x}} + x.\frac{\dot{v}}{\dot{y}}\cdot\frac{\ddot{y}}{\dot{x}^2} + x.\left(\frac{\dot{y}}{\dot{x}}\right)^2\cdot\frac{\ddot{v}}{\dot{y}^2}$$

&c. = &c.

Again, from the equation $u = f(y)$, taking $x$ as the variable, we have

$$\frac{\dot{u}}{\dot{x}} = \frac{\dot{u}}{\dot{y}} \times \frac{\dot{y}}{\dot{x}}$$

$$\frac{\ddot{u}}{\dot{x}^2} = \frac{\ddot{u}}{\dot{y}^2} \times \frac{\dot{y}^2}{\dot{x}^2} + \frac{\ddot{y}}{\dot{x}^2} \times \frac{\dot{u}}{\dot{y}}$$

&c. = &c.

But by Maclaurin's Theorem (see page 281),

$$u = U + U_1 x + U_2 . \frac{x^2}{2} + U_3 . \frac{x^3}{2.3} + \&c.$$

and $U = f(a)$, the value of $u$ when $x = 0$.

$$U_1 = \frac{(fa)^{\cdot}}{\dot{a}} \times \varphi a,$$

$$U_2 = \frac{\{(\varphi a)^2\}^{\cdot}}{\dot{a}} . \frac{(fa)^{\cdot}}{\dot{a}} + (\varphi a)^2 . \frac{(fa)^{\cdot\cdot}}{\dot{a}^2}$$

$$= \frac{\left\{(\varphi a)^2 \times \frac{(fa)^{\cdot}}{\dot{a}}\right\}^{\cdot}}{\dot{a}}$$

&c. = &c.

$$\therefore u = fa + x\varphi a . \frac{(fa)^{\cdot}}{\dot{a}} + \frac{x^2}{1.2}\frac{\left\{(\varphi a)^2 . \frac{(fa)^{\cdot}}{\dot{a}}\right\}^{\cdot}}{\dot{a}}$$

$$+ \frac{x^3}{1.2.3}\frac{\left\{(\varphi a)^3 \times \frac{(fa)^{\cdot}}{\dot{a}}\right\}^{\cdot\cdot}}{\dot{a}^2}$$

$$+ \&c.$$

This theorem (making $x = 1$) was first given by

Lagrange, whose name it bears, and is highly important. A few examples will show its advantages.

Ex. (1). *Let it be required to revert* $A+By+Cy^2+$ ........ $=0$, *or to express* $y$ *in terms of the constants* $A$, $B$, $C$, &c.

Since $y=-\frac{A}{B}-\frac{y^2}{B}.(C+Dy+Ey^2+\ldots\ldots)$, comparing it with $u=fy=f.(a+x\varphi y)$, we have

$$u=y,\ a=-\frac{A}{B},\ fa=a,\ x=1$$

$\varphi y=-\frac{y^2}{B}.(C+Dy+Ey^2+\ldots\ldots)$, and $\therefore$

$\varphi a=-\frac{a^2}{B}.(C+Da+Ea^2+\ldots\ldots)$.

Hence also $\frac{(fa)^{\cdot}}{a^{\cdot}}=\frac{a^{\cdot}}{a^{\cdot}}=1$

$$\frac{\left\{(\varphi a)^2.\frac{(fa)^{\cdot}}{a^{\cdot}}\right\}^{\cdot}}{a^{\cdot}}=\frac{2\varphi a.(\varphi a)^{\cdot}}{a^{\cdot}}=-\frac{2a^2}{B}.(C+Da+\ldots)\times\left\{\frac{-2a}{B}.(C+Da+\ldots\ldots)-\frac{a^2}{B}.(D+2Ea+\ldots).\right\}=\text{\&c. \&c. \&c.}$$

Hence

$$\begin{aligned}y=a&-\frac{Ca^2}{B}-\frac{Da^3}{B}-\frac{Ea^4}{B}-\text{\&c.}\\&+\frac{2C^2a^3}{B^2}+\frac{5CDa^4}{B^2}+\text{\&c.}\\&-\frac{5C^3a^4}{B^3}-\text{\&c.}\\&+\text{\&c.}\end{aligned}$$

where $a=-\frac{A}{B}$.

**Ex. 2.** *Let it be required to express the Eccentric Anomaly of a Planet, reckoning from the Perihelion, in Terms proceeding according to the Ascending Powers of the Eccentricity of the Orbit.*

If $y$ denote the eccentric anomaly, $e$ the eccentricity, and $nt$ the mean anomaly, it is well known that

$$y = nt + e \,.\, sin.\, y;$$

which being compared with

$$u = fy = a + x \,.\, \varphi y, \text{ we have}$$

$$u = y,\ a = nt,\ x = e,\ \varphi y = sin.\, y, \text{ and } \therefore$$

$$fa = a = nt,\ \varphi a = sin.\, a = sin.\, nt.$$

Hence $\dfrac{(fa)^{\cdot}}{a^{\cdot}} = \dfrac{nt^{\cdot}}{t^{\cdot}} = n,$

$$\frac{\left\{(\varphi a)^2 .\ \dfrac{(fa)^{\cdot}}{a^{\cdot}}\right\}^{\cdot}}{a^{\cdot}} = \frac{2n\, sin.\, nt \,.\, cos.\, nt \times nt^{\cdot} \times n}{nt^{\cdot}} =$$

$$n^2 \times sin.\, 2nt$$

$$\&c. = \&c.$$

By performing the successive operations, and substituting, we shall finally obtain

$$y = nt + e \,.\, sin.\, nt + \frac{e^2}{1.2.\ 2} \times 2 \,.\, sin.\, 2nt + \frac{e^3}{1.2.3.\ 2^2}$$

$$\times\, (3^2 .\, sin.\, 3nt - 3\, sin.\, nt) + \frac{e^4}{1.2.3.4.\ 2^3} \,.\, (4^3\, sin.\, 4nt$$

$$-\, 4 \,.\, 2^3 .\ sin.\ 2nt) + \frac{e^5}{1.2.3.4.5.\ 2^4} \times (5^4 .\, sin.\, 5nt$$

$$-\, 5 . 3^4 .\ sin.\ 3nt + .\, \frac{5.4}{1.2}\ sin.\ nt) + \&c.$$

From the comparative smallness of $e$, this series converges very rapidly.

For other examples to this theorem, see Lagrange, *Résolution des Equations Numériques*, and *A Collection* *of Examples of the Applications of the Differential and Integral Calculus.* By G. Peacock, A. M. &c.

This theorem has been extended by *Laplace* to the form

$$u = f(y) = f.\,\{f(a + x\varphi y)\},$$

*which it is required to expand according to ascending powers of x.*

Assume $u = f.\,\{f.\,(a+x\varphi y)\} = \psi.\,(a+x\varphi y)$.

Then, by Taylor's Theorem

$$u = \psi a + \frac{(\psi a)^{\cdot}}{a^{\cdot}}\,.\,x\varphi y + \frac{(\psi a)^{\cdot\cdot}}{a^{\cdot 2}}\,\frac{x^2(\varphi y)^2}{1.2} + \frac{(\psi a)^{\cdot\cdot\cdot}}{a^{\cdot 3}} \times \frac{x^3(\varphi y)^3}{1.2.3} + \&c.$$

Also $\varphi y = \varphi.\,\{f(a+x\varphi y)\} = \varphi_1\,.\,(a+x\varphi y)$ by supposition.

$$\therefore \varphi y = \varphi_1 a + \frac{(\varphi_1 a)^{\cdot}}{a^{\cdot}}\,x\,.\,\varphi y + \frac{(\varphi_1 a)^{\cdot\cdot}}{a^{\cdot 2}}\,.\,\frac{x^2(\varphi y)^2}{1.2} + \frac{(\varphi_1 a)^{\cdot\cdot\cdot}}{a^{\cdot 3}}\,\frac{x^3(\varphi y)^3}{1.2.3} + \&c.$$

$$\text{Let } \psi a = a_0,\ \frac{(\psi a)^{\cdot}}{a^{\cdot}} = a_1,\ \frac{(\psi a)^{\cdot\cdot}}{a^{\cdot 2}} = a_2,\ \&c. = \&c.$$

$$\varphi_1 a = A_0,\ \frac{(\varphi_1 a)^{\cdot}}{a^{\cdot}} = A_1,\ \frac{(\varphi_1 a)^{\cdot\cdot}}{a^{\cdot 2}} = A_2, \&c. = \&c.$$

$$\text{and } \varphi y = v.$$

$$\text{Hence } u = a_0 + a_1 v \times x + a_2 v^2 \times \frac{x^2}{1.2} + a_3 v^3 \times \frac{x^3}{1.2.3} + \&c.$$

$$\text{and } v = A_0 + A_1 v \times x + A_2 v^2 \times \frac{x^2}{1.2} + A_3 v^3 \times \frac{x^3}{1.2.3} + \&c.$$

Also $v^2 = A_0^2 + 2A_0 A_1 v \times x + \&c.$

$$v^3 = A_0^3 + \&c.$$

$$\&c. = \&c.$$

$\therefore$ by substitution, we get

$$u = a_0 + a_1 A_0 \times x + (2a_1 A_1 v + a_2 A_0^2)\frac{x^2}{1.2} +$$

$$(3a_1 A_2 v^2 + 6a_2 A_0 A_1 v + a_3 A_0^3) . \frac{x^3}{1.2.3} + \&c.$$

But, by Maclaurin's Theorem,

$$u = U + U_1 . x + U_2 . \frac{x^2}{1.2} + \&c. \text{ [see page 281.]}$$

and supposing $x = 0$ in each of the above coefficients, we get (since on that supposition $v = \phi y = \phi_1 (a + x\phi y)$ becomes $\phi_1 a$)

$$U = a_0 = \psi a$$

$$U_1 = a_1 A_0 = \frac{(\psi a)^{\cdot}}{a^{\cdot}} . \phi_1 a$$

$$U_2 = 2a_1 A_1 v + a_2 A_0^2 = 2\frac{(\psi a)^{\cdot}}{a^{\cdot}}\,\frac{(\phi_1 a)^{\cdot}}{a^{\cdot}} . \phi_1 a + \frac{(\psi a)^{\cdot\cdot}}{a^{\cdot 2}}(\phi_1 a)^2$$

$$= \frac{1}{a^{\cdot}} \times \left\{ (\phi_1 a)^2 . \frac{(\psi a)^{\cdot}}{a^{\cdot}} \right\}^{\cdot}$$

similarly $U_3 = \dfrac{1}{a^{\cdot 2}} \times \left\{ (\phi_1 a)^3 . \dfrac{(\psi a)^{\cdot}}{a^{\cdot}} \right\}^{\cdot\cdot}$

$$\&c. = \&c.$$

$$\therefore u = f . \{ f(a + x\phi y) \} = \psi . (a + x . \phi y)$$

$$= \psi a + \phi_1 a . \frac{(\psi a)^{\cdot}}{a^{\cdot}} . \frac{x}{1} + \frac{\left\{ (\phi_1 a)^2 . \frac{(\psi a)^{\cdot}}{a^{\cdot}} \right\}^{\cdot}}{a^{\cdot}} \times$$

$$\frac{x^2}{1.2} + \frac{\left\{ (\phi_1 a)^3 . \frac{(\psi a)^{\cdot}}{a^{\cdot}} \right\}^{\cdot\cdot}}{a^{\cdot 2}} \times \frac{x^3}{1.2.3} + \&c.$$

which is called *Laplace's Theorem.*

This Theorem might have been established in the same manner as Lagrange's, and *vice versâ*. The student is advised, for the sake of practice, to use both modes of demonstration.

For the application of these principles, we refer the student to the *Collection of Examples*, &c. by G. Peacock, A. M. &c., where he will find abundant illustration of this subject, especially in what relates to the investigation of the roots of equations. See also Laplace, *Méchanique Céleste, tome* 1, p. 170—181.

We now come to *Partial Fluxions*, called by D'Alembert, who applied them so successfully in many physical problems of the most arduous solution, *Partial Differences*.

If $u$ be a function of $x$ and $y$, denoted by

$$u = f.(x, y),$$

and its fluxion be first taken on the supposition that $x$ alone is variable, and on the supposition that $y$ alone is variable, we shall have results of the forms

$$F(x,y) . \dot{x}, \quad F'(x,y)\,\dot{y}:$$

which are termed the *partial fluxions of* $u$, *relative to* $x$ *and* $y$ *respectively.*

Also, since it can make no difference in the entire fluxion of $u$, whether we suppose its parts due to the variation of $x$ and $y$ respectively, to be generated together or separately, we have

$$\dot{u} = F(x,y)\,\dot{x} + F'(x,y)\,\dot{y}:$$

which Geometers represent by

$$\dot{u} = \frac{\dot{u}}{\dot{x}}\,\dot{x} + \frac{\dot{u}}{\dot{y}}\,.\,\dot{y}\ldots\ldots\ldots\ldots\ldots(a)$$

$\frac{\dot{u}}{\dot{x}}$ and $\frac{\dot{u}}{\dot{y}}$ being named the *Partial-Fluxional Co-efficients relative to* $x$ and $y$ respectively.

Again, operating in the same manner upon each of the terms in equation $(a)$, and continuing the same principle of notation, we have

$$\ddot{u} = \left(\frac{\ddot{u}}{\dot{x}^2}\,\dot{x}^2 + \frac{\ddot{u}}{\dot{x}\dot{y}}\,\dot{x}\dot{y}\right) + \left(\frac{\ddot{u}}{\dot{y}^2}\,\dot{y}^2 + \frac{\ddot{u}}{\dot{y}\dot{x}}\,\dot{y}\dot{x}\right)$$

$$= \frac{\ddot{u}}{\dot{x}^2}\,\dot{x}^2 + 2\,\frac{\ddot{u}}{\dot{x}\dot{y}}\,\dot{x}\dot{y} + \frac{\ddot{u}}{\dot{y}^2}\,\dot{y}^2 \ldots\ldots\ldots\ldots (b)$$

for $\frac{\ddot{u}}{\dot{y}\dot{x}}\,\dot{y}\dot{x} = \frac{\ddot{u}}{\dot{x}\dot{y}}\,\dot{x}\dot{y}$, since we shall obviously get the same result whether we take the fluxion of $u$, supposing $x$ constant, and then again supposing $y$ constant, or in the inverse order.

In the same manner, since, for the above reasons,

$$\left.\begin{array}{l} \dfrac{\dddot{u}}{\dot{y}^2\dot{x}} = \dfrac{\dddot{u}}{\dot{x}\dot{y}^2} = \dfrac{\dddot{u}}{\dot{y}\dot{x}\dot{y}} \\ \dfrac{\ddddot{u}}{\dot{y}^3\dot{x}} = \dfrac{\ddddot{u}}{\dot{y}\dot{x}^3} = \dfrac{\ddddot{u}}{\dot{y}^2\dot{x}^2} = \dfrac{\ddddot{u}}{\dot{x}\dot{y}\dot{x}\dot{y}} \text{ \&c.} \\ \text{\&c.} = \text{\&c.} \end{array}\right\} (m)$$

the order of the variables in the denominator showing that in which they are considered variable, we have

$$\dddot{u} = \frac{\dddot{u}}{\dot{x}^3}\,\dot{x}^3 + 3\,\frac{\dddot{u}}{\dot{x}^2\dot{y}} + 3\,\frac{\dddot{u}}{\dot{x}\dot{y}^2} + \frac{\dddot{u}}{\dot{y}^3} \ldots\ldots (c)$$

&c. = &c.

By examining the results marked *a*, *b*, *c*, the law of continuation appears very similar to that in the Binomial Theorem.

Those equations also exhibit the successive fluxions of functions of two variables in a manner different from the method used by Simpson.

Ex. Required the fluxion of $xy$.

Let $u = xy$.

Then $\dot{u} = \frac{\dot{u}}{\dot{x}}\,\dot{x} + \frac{\dot{u}}{\dot{y}} \cdot \dot{y}$, by ($a$)

but $\frac{\dot{u}}{\dot{x}}\,\dot{x} = \dot{x}y$ (considering $y$ constant) and $\frac{\dot{u}}{\dot{y}}\,\dot{y} = y\dot{x}$ ($x$ being constant)

$\therefore\ \dot{u} = y\dot{x} + x\dot{y}$. [See page 7.]

If $u = f(x, y, z, \&c.)$ the same considerations will show that

$$\dot{u} = \frac{\dot{u}}{\dot{x}}\,\dot{x} + \frac{\dot{u}}{\dot{y}}\,\dot{y} + \frac{\dot{u}}{\dot{z}}\,\dot{z} + \&c.$$

and so on for any order of fluxions.

The following Theorem of Euler affords an application of this theory.

*If $u = f(x, y, z \ldots\ldots)$ be an homogeneous function of $x, y, z \ldots\ldots$ whose terms each rise to the same dimension $m$, and we have*

$$\dot{u} = X\dot{x} + Y\dot{y} + Z\dot{z} + \ldots\ldots$$

*then it may be proved that*

$$u = \frac{Xx + Yy + Zz + \ldots\ldots\ldots}{m}.$$

By putting $y = y'x$, $z = z'x$, &c. we easily transform $u$ to

$$u = x^m.f.\left(\frac{y}{x}, \frac{z}{x}, \ldots\ldots\right)$$

and taking the fluxion relatively to each of the variables, $x^m, \frac{y}{x}, \frac{z}{x} \ldots\ldots$ by the principles just delivered,

$$\left.\begin{array}{l} \dot{u} = mx^{m-1}\dot{x}.f\left(\frac{y}{x}, \frac{z}{x}, \ldots\ldots\right) \\ + x^m \times \frac{\dot{u}}{\left(\frac{y}{x}\right)^{\cdot}}.\frac{x\dot{y} - y\dot{x}}{x^2} \\ + x^m \times \frac{\dot{u}}{\left(\frac{z}{x}\right)^{\cdot}}.\frac{x\dot{z} - z\dot{x}}{x^2} \\ + \&c \end{array}\right\} =$$

$$\{mx^{m-1}f\left(\frac{y}{x}, \frac{z}{x}......\right) - x^{m-2}y \,.\, P - x^{m-2}zQ - ......\}\dot{x}$$

$$+ x^{m-1}P\dot{y} + x^{m-1}Q\dot{z} + .........$$

$P$, $Q$, &c. being put $= \frac{\dot{u}}{\left(\frac{y}{x}\right)^{\cdot}}, \; \frac{\dot{u}}{\left(\frac{z}{x}\right)^{\cdot}}$. &c.

But $\dot{u}$ also $= X\dot{x} + Y\dot{y} + Z\dot{z} + .........$

$\therefore$ equating co-efficients of $\dot{x}$, $\dot{y}$, &c. we get

$$X = mx^{m-1}f \cdot \left(\frac{y}{x}, \frac{z}{x} ......\right) - x^{m-2}yP - x^{m-2}zQ$$

$$- \text{\&c.}, \; Y = x^{m-1}P, \; Z = x^{m-1}Q, \; \text{\&c.} = \text{\&c.}$$

and substituting for $f\left(\frac{y}{x}, \frac{z}{x} ......\right)$, $P$, $Q$, &c. transposing the negative terms, and dividing by $\frac{m}{x}$, there finally results

$$mu = Xx + Yy + Zz + .........$$

This theorem is useful in the integration of homogeneous equations.

We will give another example.

*Given two different functions of $x$, $y$*

$$F(x, y) \text{ and } f.\,(x, y),$$

*to determine if one function be a function of the other.*

Let $u = f\,(x, y)$ ... $(e)$, and assume

$$F(x, y) = \phi\,\{f\,(x, y)\} = \phi\,(u) = v.$$

Hence $\frac{\dot{v}}{\dot{x}}\,\dot{x} + \frac{\dot{v}}{\dot{y}}\,\dot{y} = \dot{v} = \dot{u}\,\phi'(u)$ by *hyp.*

But from equation $(e)$

$$\dot{u} = \frac{\dot{u}}{\dot{x}}\,\dot{x} + \frac{\dot{u}}{\dot{y}}\,\dot{y}$$

$$\therefore \frac{\dot{v}}{\dot{x}} \dot{x} + \frac{\dot{v}}{\dot{y}} \dot{y} = \frac{\dot{u}}{\dot{x}} \dot{x} \times \varphi' u + \frac{\dot{u}}{\dot{y}} \dot{y} \, \varphi'(u)$$

which can be true only when

$$\frac{\dot{v}}{\dot{x}} = \frac{\dot{u}}{\dot{x}} \varphi'(u), \text{ and } \frac{\dot{v}}{\dot{y}} = \frac{\dot{u}}{\dot{y}} \varphi'(u)$$

$$\text{Hence } \frac{\dot{v}}{\dot{x}} \cdot \frac{\dot{u}}{\dot{y}} = \frac{\dot{v}}{\dot{y}} \cdot \frac{\dot{u}}{\dot{x}} \ \ldots\ldots\ldots \ (f)$$

the equation of condition, which being satisfied

$F(x, y)$ is a function of $f(x, y)$.

As a particular instance, take

$$\left.\begin{array}{l} (a^3x^3 + b^3y^3)(ax + by)^{-1} = F(x, y) \\ \text{and } a^2x^2 + b^2y^2 - abxy = f \cdot (x, y). \end{array}\right\}$$

$$\text{Here } \frac{\dot{v}}{\dot{x}} = \frac{2a^4x^3 + 3a^3byx^2 - ab^3y^3}{(ax + by)^2}$$

$$\frac{\dot{v}}{\dot{y}} = \frac{2b^4y^3 + 3ab^3xy^2 - a^3bx^3}{(ax + by)^2}$$

$$\frac{\dot{u}}{\dot{x}} = 2a^2x - aby$$

$$\frac{\dot{u}}{\dot{y}} = 2b^2y - abx$$

which being substituted in equation $(f)$ the two members are found to be identical, and $F(x, y)$ is therefore a function of $f(x, y)$.

It is easily seen indeed that

$$F(x, y) = (ax + by) \times f(x, y).$$

In this place we might show the method of eliminating constants from equations between two or more variables, by successively taking the fluxions and making proper substitutions; but as it involves no new principles, nor any difficulty in the application of those already explained, we shall proceed to matters of greater importance.

We shall here treat of *Vanishing Fractions*, or those fractions which, under certain circumstances, assume the indeterminate form $\frac{0}{0}$.

Let $\frac{N}{D} = Q$ be such that ($N$ and $D$ are functions of $x$) $x$ being put $= a$, both numerator and denominator may vanish; required under these circumstances, a determinate value of $\frac{N}{D}$.

Let $N_a$, $D_a$, $Q_a$, &c. denote the values of $N$, $D$, $Q$, &c. when for $x$ we have substituted $a$.

Then since $N = DQ$

$$N^{\cdot} = D^{\cdot}Q + Q^{\cdot}D$$

$$\therefore \frac{N^{\cdot}}{D^{\cdot}} = Q + \frac{Q^{\cdot}D}{D^{\cdot}}$$

$$\text{and } \frac{N^{\cdot}_a}{D^{\cdot}_a} = Q_a = \frac{N_a}{D_a}, \text{ since } D_a = 0.$$

Again, if $\frac{N^{\cdot}_a}{D^{\cdot}_a}$ also $= \frac{0}{0}$, we have

$$N^{\cdot\cdot} = D^{\cdot\cdot}Q + 2Q^{\cdot}D^{\cdot} + Q^{\cdot\cdot}D$$

$$\therefore \frac{N^{\cdot\cdot}}{D^{\cdot\cdot}} = Q + \frac{2\,Q^{\cdot}D^{\cdot}}{D^{\cdot\cdot}} + \frac{Q^{\cdot\cdot}}{D^{\cdot\cdot}}D$$

$$\text{and } \frac{N^{\cdot\cdot}_a}{D^{\cdot\cdot}_a} = Q_a = \frac{N_a}{D_a}, \text{ since } D^{\cdot} \text{ and } D \text{ each} = 0.$$

If $\frac{N^{\cdot\cdot}_a}{D^{\cdot\cdot}_a}$ still $= \frac{0}{0}$, we must repeat the operation, and so on, till at length we arrive at a determinate value. This value will be that of $\frac{N_a}{D_a} = \frac{0}{0}$. [See page 155.]

Ex. 1. *Let* $Q = \frac{a - \sqrt{a^2 - x^2}}{x^2}$

Then $Q_a = \dfrac{N^{\cdot\cdot}_a}{D^{\cdot\cdot}_a} = \dfrac{1}{2a}$.

Ex. 2. *Let* $Q = \dfrac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\sqrt{2ax - x^2}}{x^2 - 2ax - a^2 + 2a\sqrt{2ax - x^2}}$

Then $Q_a = \dfrac{\overset{::}{N}_a}{\overset{::}{D}_a} = -\, 5a.$

Ex 3. *Let* $Q = \dfrac{a^n - x^n}{la - lx}$ ($l$ denoting the hyperbolic logarithm.)

Then $Q_a = \dfrac{N^{\cdot}_a}{D^{\cdot}_a} = na^n.$

Ex. 4. *Let* $Q = \dfrac{1 - \dfrac{2x}{\pi}}{cot.\ x}$.

Then $Q_{\left(\frac{\pi}{2}\right)} = \dfrac{N^{\cdot}_{\left(\frac{\pi}{2}\right)}}{D^{\cdot}_{\left(\frac{\pi}{2}\right)}} = \dfrac{2}{\pi}.$

Ex. 5.

*Let* $Q = \dfrac{x + x^2 - (n+1)^2x^{n+1} + (2n^2 + 2n - 1)\, x^{n+2} - n^2x^{n+3}}{(1-x)^3}$

Then $Q_1 = \dfrac{\overset{\therefore}{N}_1}{\overset{\therefore}{D}_1} = \dfrac{n\, .\, (n+1)\ 2n + 1)}{1.2.3}$ which expresses the sum of the series

$$1 + 2^2 + 3^2 + \ldots\ldots\ n^2.$$

This method failing in some cases of $N$ and $D$ being functions of irrational binomials, trinomials, &c. we shall subjoin a process of general application, which has the advantage also of being very simple.

Let $a+h$ be substituted for $x$ in $N$ and $D$, which let be expanded by Taylor's theorem, or any other equivalent process, in series ascending according to the powers of $h$, so that we may have

$Q_{a+h}=\dfrac{Ah^m+Bh^n+\ldots\ldots}{A'h^{m'}+B'h^{n'}+\ldots\ldots}$ which being considered according to the three cases $m >, =$, or $< m'$, will give, by putting $h=0$,

$$Q_a = 0$$

$$\text{or } Q_a = \frac{A}{A'}$$

$$\text{or } Q_a = \frac{1}{0} = \infty.$$

Hence, then, the following general rule for finding the values of all functions of $x$, which assume the form $\frac{0}{0}$ when $x=a$.

*Take the first term of $N$ and $D$ expanded according to the ascending powers of $h$ ($x$ being put $=a+h$), reduce the resulting fraction to its lowest terms, and then put $h=0$: the result will express the value required.*

We thus find $\dfrac{\pi \,.\, \sin.\, \frac{\pi x}{2}}{4x \cos.\, \frac{\pi x}{2}}$ becomes $\frac{0}{0}=\frac{\pi^2}{8}$, on the supposition that $x=0$.

Also that $\dfrac{\pi x e^{2\pi x}-e^{2\pi x}+\pi x+1}{2x^2 e^{2\pi x}-2x^2}$ becomes $\frac{0}{0}=\frac{\pi^2}{6}$ when $x=0$; which functions express the series

$$\frac{1}{1-x^2}+\frac{1}{3^2-x^2}+\frac{1}{5^2-x^2}+\ldots\ldots\ldots\infty$$

and $\dfrac{1}{1^2+x^2}+\dfrac{1}{2^2+x^2}+\dfrac{1}{3^2+x^2}+\ldots\ldots\ldots\infty$

respectively.

Sometimes $\frac{N_a}{D_a}$ is of the form $\frac{\infty}{\infty}$ which may, however, be reduced to $\frac{0}{0}$, and treated accordingly.

$$\text{For } \frac{N^a}{D_a} = \frac{\infty}{\infty} = \frac{\left(\frac{1}{0}\right)}{\left(\frac{1}{0}\right)} = \frac{1}{0} \times \frac{0}{1} = \frac{0}{0}.$$

$$\text{Also we have } \frac{N_a}{D_a} = \infty - \infty = \frac{0}{0}.$$

$$\text{For } \infty - \infty = \infty \times (1 - 1) = \frac{1}{0} \times 0 = \frac{0}{0}.$$

Examples of these kinds of functions are

(1.) $\frac{N}{D} = \frac{cot. \frac{\theta}{2^x}}{2^x} - 2\, cot.\, 2\theta$, which becomes $\frac{\infty}{\infty} - 2\, cot.\, 2\theta$, when $x = \infty$; and $\frac{N_\infty}{D_\infty} = \frac{1}{\theta} - 2\, cot.\, 2\theta$, which is the value of $tan.\, \theta + \frac{1}{2}\, tan.\, \frac{\theta}{2} + \frac{1}{2^2} \cdot tan.\, \frac{\theta}{2^2}$ + &c. to $\infty$.

(2.) $\frac{N}{D} = \frac{\pi}{4x} + \frac{\pi}{2x\,(e^{\pi x} - 1)}$, which expresses the series

$$\frac{1}{1 + x^2} + \frac{1}{3^2 + x^2} + \ldots\ldots \infty$$

$$\text{Here } \frac{N_0}{D_0} = \infty - \infty = \frac{0}{0} = \frac{\pi^2}{8}.$$

SECT. II.—The subject of *maxima* and *minima* has been so amply and ably treated by our author, that little need be given in addition to this section. We shall content ourselves with applying Taylor's theorem to distinguish between *maxima* and *minima* of functions of one variable, and a few examples in Transcendental Functions (exponential, logarithmic, circular, &c.), and in functions of two or more independent variables.

*The function of* $x$ *expressed by* $u=f(x)$ *admits of a maximum or minimum, if* $\frac{\dot{u}}{\dot{x}}=0$, *or if each of* $\frac{\dot{u}}{\dot{x}}$, $\frac{\ddot{u}}{\dot{x}^2}$ and $\frac{\dddot{u}}{\dot{x}^3}=0$, *or if each of* $\frac{\dot{u}}{\dot{x}}$, $\frac{\ddot{u}}{\dot{x}^2}$, $\frac{\dddot{u}}{\dot{x}^3}$, $\frac{\overset{4}{\dot{u}}}{\dot{x}^4}$, and $\frac{\overset{5}{\dot{u}}}{\dot{x}^5}$ $=0$, or &c. according as $\frac{\ddot{u}}{\dot{x}^2}$, or $\frac{\ddddot{u}}{\dot{x}^4}$, or $\frac{\overset{6}{\dot{u}}}{\dot{x}^6}$, or &c. respectively, is negative or positive.

For, by Taylor's theorem [see page 284.]

$$u_1=f(x-h)=u-\frac{\dot{u}}{\dot{x}}h+\frac{\ddot{u}}{\dot{x}^2}\cdot\frac{h^2}{1.2}-\ldots\ldots\ldots$$

$$u_2=f.(x+h)=u+\frac{\dot{u}}{\dot{x}}h+\frac{\ddot{u}}{\dot{x}^2}\cdot\frac{h^2}{1.2}+\ldots\ldots\ldots$$

$u_1$ and $u_2$ being the values immediately preceding and succeeding $u$.

Now when $h$ is very small, any one term of the series $Ah+Bh^2+\ldots\ldots\infty$ is greater than the sum of all the succeeding terms: for $Bh^2+Ch^3+\ldots\ldots=h^2.(B+Ch+Dh^2+\ldots\ldots)$, and $Ch+Dh^2+\ldots\ldots=h.(C+Dh+\ldots\ldots)$ is evanescent compared with $B$, $\therefore$ $Bh^2$ is greater than $Ch^3+Dh^4+\ldots\ldots$ and so on

Hence all the terms beyond $\frac{\ddot{u}}{\dot{x}^2}\cdot\frac{h^2}{1.2}$ in $u_1$, $u_2$ may be neglected, and we have

$$u_1 = u - \frac{\dot{u}}{\dot{x}} h + \frac{\ddot{u}}{\dot{x}^2} \cdot \frac{h^2}{1.2}$$

$$u_2 = u + \frac{\dot{u}}{\dot{x}} h + \frac{\ddot{u}}{\dot{x}^2} \cdot \frac{h^2}{1.2}$$

But in the case of $u = max.$ $u$ is greater than both $u_1$ and $u_2$, which cannot possibly be unless $\frac{\dot{u}}{\dot{x}} = 0$, and $\frac{\ddot{u}}{\dot{x}^2}$ is negative. And when $u = min.$ $u$ is less than either $u_1$ or $u_2$; therefore, in this case, $\frac{\dot{u}}{\dot{x}} = 0$, and $\frac{\ddot{u}}{\dot{x}^2}$ is positive.

Again, let both $\frac{\dot{u}}{\dot{x}}$ and $\frac{\ddot{u}}{\dot{x}^2}$ equal 0.

$$\text{Then } \left.\begin{aligned} u_1 &= u - \frac{\dddot{u}}{\dot{x}^3} \cdot \frac{h^3}{1.2.3} + \frac{\ddddot{u}}{\dot{x}^4} \cdot \frac{h^4}{1.2.3.4} \\ u_2 &= u + \frac{\dddot{u}}{\dot{x}^3} \cdot \frac{h^3}{1.2.3} + \frac{\ddddot{u}}{\dot{x}^4} \cdot \frac{h^4}{1.2.3.4} \end{aligned}\right\} \text{the}$$

higher powers of $h$ being neglected for the same reason as before. By the above mode of reasoning, it also appears in this case, when $u = max.$ or $min.$ we have

$$\frac{\dot{u}}{\dot{x}} = 0, \ \frac{\ddot{u}}{\dot{x}^2} = 0, \ \frac{\dddot{u}}{\dot{x}^3} = 0, \text{ and } \frac{\ddddot{u}}{\dot{x}^4} = \text{a negative or}$$

positive quantity, according as $u = max.$ or $min.$ The same process may evidently be continued.

In functions of two or more variables, the characters which distinguish the *maxima* from the *minima*, may, by a similar but more complicated process, be ascertained.

If $u = f(x, y)$, it will be found that according as $u = max.$ or $min.$ the *partial fluxions*

$$\left.\begin{array}{r}\frac{\ddot{u}}{\dot{x}^2}\,\dot{x}^2\\ \text{and }\frac{\ddot{u}}{\dot{y}^2}\,\dot{y}^2\end{array}\right\}\text{ are both negative, or both positive, and}$$

in both cases,

$$\frac{\ddot{u}}{\dot{x}^2}\,\dot{x}^2 \times \frac{\ddot{u}}{\dot{y}^2} \times \dot{y}^2 \text{ is greater than } \left(\frac{\ddot{u}}{\dot{x}\dot{y}}\right)^2.$$

## Examples in Transcendental Functions.

1. *To find that Number which bears the greatest Ratio to its Logarithm (hyperbolic).*

Let $x$ be the number.

Then $\frac{x}{lx} = max.$ and taking the fluxions, we have

$$\frac{\dot{x}}{lx} - \frac{\dot{x}}{(lx)^2} = 0 \text{ [see page 277.]}$$

$\therefore lx = 1 = le$, or $x = e$ the hyperbolic base.

(2). *To divide a Number n into so many Parts, that their continued Product may be a Maximum.*

Let $x$ be the number of parts.

Then $\left(\frac{n}{x}\right)^x = max.$

and since the greater a quantity is, the greater is its logarithm to the same base.

$$l.\left(\frac{n}{x}\right)^x = x\,.\,l\frac{n}{x} = max.$$

$$\therefore \dot{x}\,.\,l\frac{n}{x} - n\dot{x} = 0$$

$$\therefore l\frac{n}{x} = 1 = l\,.\,e.$$

$$\therefore \frac{n}{x} = e, \text{ and } x = \frac{n}{e}.$$

Also $e^{\frac{n}{e}}$ = the *maximum.*

(3). *To divide an Arc $A$ (Radius of the Circle being 1) into two such Parts, that the Product of the $(n)^{th}$ Power of the one, and the $m^{th}$ Power of the other, may be a Maximum.*

Let $\theta$ = one part.

Then $A - \theta$ = the other, and

$$sin.^{n}\theta \times sin.^{m}(A - \theta) = max.$$

and taking the logarithms

$$n\,l.\,sin.\,\theta + m\,l.\,sin.\,(A - \theta) = max.$$

$$\therefore \frac{n\,\theta^{\cdot}\,cos.\,\theta}{sin.\,\theta} - \frac{m\,\theta^{\cdot}\,cos.\,(A - \theta)}{sin.\,(A - \theta)} = 0.$$

Hence $\dfrac{m}{n} = \dfrac{tan.\,(A - \theta)}{tan.\,\theta}$

and $\dfrac{m + n}{m - n} = \dfrac{tan.\,(A - \theta) + tan.\,\theta}{tan.\,(A - \theta) - tan.\,\theta} = \dfrac{sin.\,A}{sin.(A - 2\theta)}$

$\therefore sin.\,(A - 2\theta) = \dfrac{m - n}{m + n}\,.\,sin.\,A$, which gives $A - 2\theta$, and $\therefore \theta$. Hence also $A - \theta$ is known.

4. *Required the value of $x$ when $\dfrac{e^{x}}{sin.(a - x)}$ = a minimum.*

$$Answer\ x = a + \frac{\pi}{4}.$$

The following examples will illustrate Art. 45, p. 43.

(1.) *Amongst all Parallelopepedons, whose Planes are perpendicular to one another, and of given Magnitude, find that which has the least Surface.*

Let $x$, $y$, $z$, be the three edges of the parallelopepedon.

Then, since the magnitude is given

$$x\,y\,z = a \text{ (a given quantity)}$$

$$\text{and } 2xy + 2xz + 2yz = a\ minimum$$

∴ by substituting for $x$, and taking the half,

$$xy + x \times \frac{a}{xy} + y . \frac{a}{xy} = min.$$

$$\text{or } u = xy + \frac{a}{y} + \frac{a}{x} = min.$$

$$\left.\begin{aligned} \therefore \frac{\dot{u}}{\dot{x}} &= y - \frac{a}{x^2} = 0 \\ \text{and } \frac{\dot{u}}{\dot{y}} &= x - \frac{a}{y^2} = 0 \end{aligned}\right\} \text{see page 43.}$$

$$\text{Hence } y = \frac{a}{x^2} = a \times \frac{y^4}{a^2} = \frac{y^4}{a}.$$

$$\left.\begin{aligned} \therefore y &= a^{\frac{1}{3}} \\ \text{also } x &= a^{\frac{1}{3}} \\ \text{and} \therefore z &= a^{\frac{1}{3}} \end{aligned}\right\} \text{which indicates that the parallelopi-}$$

pedon is a cube.

(2.) *If $a$, $b$, $c$, be the prime Factors composing a given Number $A$, required the Number of Times each must enter that Number, so that it may contain the greatest possible Number of Divisors.*

Let $x$, $y$, $z$, be the number of times $a$, $b$, $c$, are taken respectively.

Then $a^x b^y c^z = A$.

And $u = (x+1).(y+1)\ (z+1) = max.$ (see Barlow's Theory of Numbers, p. 32).

Now first we will eliminate $z$ by means of

$$xl.a + yl.b + zlc = lA$$

$$\text{whence } u = (x+1).(y+1)\left(\frac{lA}{lc} - \frac{xla}{lc} - y\frac{lb}{lc} + 1\right)$$

$$\left.\begin{aligned}
&\therefore \frac{\dot{u}}{\dot{x}} = (y+1).\left(\frac{lA}{lc} - \frac{xla}{lc} - y.\frac{lb}{lc} + 1\right) \\
&- (x+1)\,(y+1)\,\frac{la}{lc} = 0 \text{ (by Art. 45).} \\
&\text{and } \frac{\dot{u}}{\dot{y}} = (x+1).\left(\frac{lA}{lc} - \frac{xla}{lc} - \frac{ylb}{lc} + 1\right) \\
&- (x+1).(y+1).\frac{lb}{lc} = 0.
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\text{Hence } lA - xla - ylb + lc &= (x+1).la \\
\text{and } lA - xla - ylb + lc &= (y+1).lb
\end{aligned}\right\}$$

$$\therefore (x+1)\,la = (y+1).lb$$

$$\therefore xla = ylb - l.\frac{a}{b}:$$

$$\text{similarly } zlc = ylb - l.\frac{c}{b}$$

$$\text{also} = lA - ylb - xla.$$

∴ we have the two equations

$$\left.\begin{aligned}
xla &= ylb - l.\frac{a}{b} \\
\text{and } xla &= lA + l.\frac{c}{b} - 2ylb
\end{aligned}\right\} \text{which give}$$

$$3ylb = lA + l\frac{c}{b} + l.\frac{a}{b}$$

$$= l.(Aac) - 2l.b$$

$$\therefore y = \frac{l.(Aac) - 2lb}{3lb}:$$

$$\text{similarly } x = \frac{l.(Abc) - 2la}{3la},$$

$$\text{and } z = \frac{l\,(Aab) - 2lc}{3lc}.$$

**SECT. III.**—Supplementary to this section, which treats of drawing tangents to curves, we may give other modes of determining the position of the tangent, with additional examples in *transcendental* and other curves; also a brief exposition of the method of drawing *asymptotes*, or *those lines which touch the curve at a point whose abscissa is infinite.*

(1.) *If x and y denote the abscissa and ordinate of a curve respectively, and θ the inclination of the tangent to the line of abscissæ, then* $\frac{\dot{y}}{\dot{x}} = tan.\ \theta$.

For the angle $SCn = CFm = \theta$ (fig. page 54), and $\dot{y} : \dot{x} :: Sn : Cn :: 1 : tan.\ \theta$

$$\therefore \frac{\dot{y}}{\dot{x}} = tan.\ \theta \ldots\ldots\ldots\ldots (a)$$

This equation is useful in determining the position of the tangent, or of the curve, at particular points in the curve; *e. g. required the angle at which a circle cuts its diameter.*

Here $y^2 = 2rx - x^2$.

$$\therefore tan.\ \theta = \frac{\dot{y}}{\dot{x}} = \frac{r-x}{\sqrt{2rx - x^2}} = \frac{r-x}{y}$$

Let $x$ and therefore $y = 0$.
Then, at the point of intersection,

$$tan.\ \theta = \frac{r}{0} = \infty,$$

$$\text{or } \theta = 90^\circ,$$

which we also know from *common geometry.*

(2.) *Required the Conditions of Contact of any two Curves whose Equations are* $y = f.(x)$, $\acute{y} = f'(\acute{x})$, *x and x́ being measured on the same straight line.*

Since at the point of contact the two curves meet, we there have

$$\acute{y} = y.$$

Also, the curves having the same straight line touching them both at the point of contact,

$$\frac{\dot{y}'}{\dot{x}'} = tan.\ \theta \text{ (see equation } a\text{)} = \frac{\dot{y}}{\dot{x}}.$$

The conditions of contact are, therefore,

$$\left.\begin{array}{l} y' = y \\ \text{and } \dfrac{\dot{y}'}{\dot{x}'} = \dfrac{\dot{y}}{\dot{x}} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (b).$$

The use of these expressions will readily be perceived from what follows.

Ex. 1. *Required to draw a Parabola whose Axis shall coincide with that of a given Circle, and touch the Circle in a given Point.*

$$\left.\begin{array}{l} \text{Let } y^2 = 2rx - x^2 \\ \text{and } y'^2 = px' \end{array}\right\} \text{ be the equations of the circle}$$

and parabola, $p$ (the *latus rectum* of the parabola) being, at present, undetermined.

$$\text{Then } \frac{\dot{y}'}{\dot{x}'} = \frac{p}{2y'}.$$

$$\text{and } \frac{\dot{y}}{\dot{x}} = \frac{r-x}{y}$$

Hence, at the point of contact,

$$\frac{p}{2y'} = \frac{r-x}{y} \text{ (by equation } b\text{)}$$

$$\text{or } p = \frac{y'}{y} \times 2 \times (r-x) = 2.(r-x)$$

$$\therefore x' = \frac{y'^2}{p} = \frac{y^2}{p} = \frac{2rx - x^2}{2.(r-x)}.$$

If, therefore, a parabola, whose latus rectum is $2.(r-x)$, and vertex, distant from the ordinate of the given point in the circle, by $\acute{x} = \dfrac{2rx - x^2}{2\,(r-x)}$, be described, it shall touch the circle at the given point.

Ex. 2. *To draw an Ellipse of given Eccentricity, having the same line of abscissæ as a given Circle, and touching it in a given Point.*

$$\left.\begin{array}{l} \text{Here } y^2 = 2rx - x^2 \\ \text{and } \acute{y}^2 = \dfrac{b^2}{a^2}.\,(2a\acute{x} - \acute{x}^2) \end{array}\right\} \; b \text{ and } a \text{ being the}$$

constants to be determined.

At the point of contact

$$\frac{r-x}{y} = \frac{\dot{y}}{\dot{x}} = \frac{\dot{y}'}{\dot{x}'} = \frac{b^2}{a^2}.\ \frac{a-x}{\acute{y}}$$

whence $b^2 = \dfrac{r-x}{a-x} \times a^2$.

Let the given eccentricity $= e$

Then $a^2 - e^2 = b^2 = \dfrac{r-x}{a-x} \times a^2$,

and by reduction we get

$$a^3 - ra^2 - e^2a = xe^2,$$

from which, having found the *possible* values of $a$, we get $\acute{x}$, which enables us to construct the tangent curve as in the preceding example.

Ex. 3. *To draw a Straight Line touching any given Curve in a given Point.*

$$\left.\begin{array}{l} \text{Let } y = f.\,(x) \text{ be the equation of the curve} \\ \quad \acute{y} = A + B\acute{x} \text{ that of the tangent} \end{array}\right\}$$

$A$ and $B$ being to be determined, and $x$ and $x'$ having the same origin and direction.

At the point of contact

$$B = \frac{\dot{y}'}{\dot{x}'} = \frac{\dot{y}}{\dot{x}},\ \text{and } y = \acute{y},\ \text{and } x = x'$$

$$\therefore A = y' - Bx' = y' - \frac{\dot{y}}{\dot{x}}x'$$

$$= y - \frac{\dot{y}}{\dot{x}}x$$

$$\therefore y' = y - \frac{\dot{y}}{\dot{x}}x + \frac{\dot{y}}{\dot{x}} \times x'$$

$$\text{or } y' - y = \frac{\dot{y}}{\dot{x}} \times (x' - x) \ldots\ldots\ldots\ldots\ldots\ldots (c)$$

which is the equation of the tangent to any curve, and may be used in many cases more commodiously in determining its position, than the common expression for the subtangent.

The normal being perpendicular to the tangent, will have for its equation

$$y'' - y = -\frac{\dot{x}}{\dot{y}} \times (x'' - x) \ldots\ldots\ldots (d)$$

$y''$ and $x''$ being its co-ordinates, and $x, y$, those of the curve.

Thus, in the hyperbola referred to its asymptotes, the equation being

$$y = \frac{ab}{x}$$

we have, from (c)

$$y' = y - \frac{ab}{x^2} . (x' - x) = \frac{ab}{x} - \frac{abx'}{x^2} + \frac{ab}{x} = -\frac{ab}{x^2} \times x' + \frac{2ab}{x}.$$

And from (d)

$$y'' = \frac{x^3x'' + a^2b^2 - x^4}{abx}.$$

Again, in the *Cissoid of Diocles*,

$$y^2 = \frac{x^3}{a - x} \text{ (page 59)}$$

and we readily find the equations of the tangent and normal to be

$$y' = \frac{\sqrt{x}}{2.(a-x)^{\frac{3}{2}}}.(\overline{3a-2x}.x'-ax)$$

$$\text{and } y'' = \frac{1}{\sqrt{ax-x^2}} \times \frac{ax.(2a-x)-2.(a-x)^2x''}{3a-2x}.$$

Also in the *Witch*, the equation being

$$y = \frac{a\sqrt{ax-x^2}}{x},$$

we have

$$y' = -\frac{a}{2x.\sqrt{ax-x^2}} \times (ax'+2x^2-3ax)$$

$$\text{and } y'' = \frac{2x.\sqrt{ax-x^2}}{a^2}.\left(x''-x+\frac{a^3}{2x^2}\right).$$

In the *logarithmic curve* the equation is

$$y = a^x,$$

and it easily appears that the

$$\text{subtangent} = \frac{y\dot{x}}{\dot{y}} = \frac{1}{l.a},$$

$$y' = a^x + (x'-x)\,a^x l.a,$$

$$\text{and } y'' = a^x - (x'-x) \times \frac{1}{a^x l.a}.$$

In the curve whose equation is

$$y = x^x$$

$$\text{The subtangent} = \frac{1}{1+l.x} \text{ (see page 277)}$$

$$y' = x^x + (x'-x).(1+l.x)\,x^x$$

$$\text{and } y'' = x^x - \frac{x'-x}{(1+l.x)\,x^x}.$$

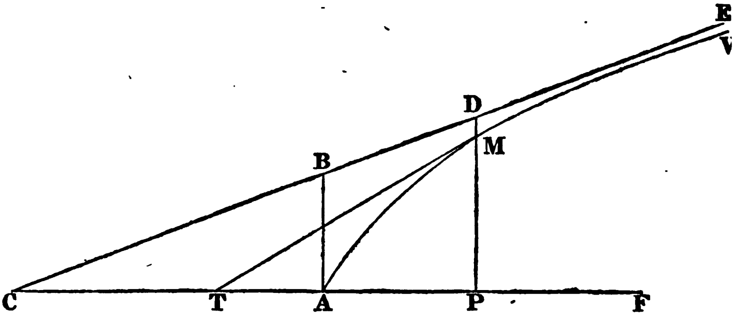
The reader will find no great difficulty in applying these forms in the solution of the Problems

*To draw a straight Line from a given Point touching a given Curve ;*

*To draw a Curve of a given nature passing through a given Point, and touching a given Curve.*

(3). We now proceed to determine the *asymptotes* (if any) of a curve defined by parallel ordinates; a subject of considerable importance, although entirely overlooked by our author.

Let C D E be an *asymptote*, or tangent at an infinite distance, to the curve A M V.



Also let M T be a tangent at the point M, and draw A B, P D, at right angles, to C F ; and put A P $= x$, P M $= y$.

Then since

$$\mathrm{PT} = \frac{y\dot{x}}{\dot{y}}, \text{ and } \mathrm{AP} = x,$$

$$\therefore \mathrm{AT} = \frac{y\dot{x}}{\dot{y}} - x \text{ which } = \mathrm{AC} \text{ when } x \text{ is infinite.}$$

Also $\frac{\dot{y}}{\dot{x}} =$ *tan.* M T P (see page 307) which $=$ *tan.* A C B $= \frac{\mathrm{AB}}{\mathrm{AC}}$, when $x$ is infinite.

Hence by making $x$ infinite in the values, deduced from the equation of the curve, of

$$\left.\begin{array}{l} \dfrac{y\dot{x}}{\dot{y}} - x \\ \text{and } \dfrac{\dot{y}}{\dot{x}} \times \left(\dfrac{y\dot{x}}{\dot{y}} - x\right) \text{ or } y - \dfrac{x\dot{y}}{\dot{x}} \end{array}\right\} \cdots\cdots\cdots (e)$$

we get A C and A B, which are sufficient to determine the position of the asymptote.

To find all the asymptotes we must treat the equations (*e*) in a similar manner for the infinite value of $y$.

If A C and B C be both finite, C B being joined and produced indefinitely, will be the asymptote required.

If A C be infinite and A B finite, C E being drawn through B at right angles to A B, or parallel to A F, will be the asymptote.

If A C be finite, and A B infinite, the asymptote will pass through C at right angles to C F.

If A C and A B be both infinite, the curve will have no asymptote.

If A C and A B both $= 0$, the asymptote passes through A, and its direction is found by $\frac{\dot{y}}{\dot{x}} = \tan.\,\theta$ (see page 307).

If $\text{AC} = 0$, and AB be infinite, it coincides with AB.

If A C be infinite, and $\text{A C} = 0$, it coincides with A C.

Ex. 1. *Let the Curve be the common Hyperbola, whose Equation is* $y^2 = \frac{b^2}{a^2} \cdot (2ax + x^2)$.

$$\text{Then P T} = \frac{2ax + x^2}{a + x} \text{ (page 58)}$$

$$\text{and} \therefore \text{A T} = \text{P T} - x = \frac{ax}{a + x}$$

$$\text{or A C} = \frac{a \times \infty}{\infty} = a.$$

Hence C is the center of the hyperbola.

Again, $y - \frac{x\dot{y}}{\dot{x}} = \frac{bx}{\sqrt{2ax + x^2}}$

$$\therefore AB = \frac{b \times \infty}{\sqrt{\infty^2}} = \pm\, b.$$

Hence, drawing AB, AB′ = ± $b$, on different sides of CA, and joining CB, &c. we get the position of the asymptotes, the same as by the ordinary methods of *geometry*.

Ex. 2. *Let the Equation of the Curve be*

$$y^3 - x^3 = axy.$$

Then it will be found that

$$AT = \frac{3y^3 - 3x^3 - 2axy}{3x^2 + ay}$$

$$= \frac{axy}{3x^2 + ay}$$

and $\therefore AC = \frac{axy}{3x^2} = \frac{ay}{3x}$ ($y$ being finite).

Now, to separate the variables, put $\frac{ay}{3x} = u$ (which will apply in similar cases). Hence $y = 3xu$, and substituting in the original equation, &c., we get

$$\frac{27u^3x}{a^3} - x - 3u = 0,$$

and $u$, being finite, may be omitted when $x$ is infinite.

$$\therefore\ 27u^3 = a^3$$

$$\text{or } u = \frac{a}{3}$$

$$\text{Hence } AC = \frac{a}{3}$$

Also $y - \frac{x\dot{y}}{\dot{x}} = \frac{axy}{3y^2 - ax} = \frac{axy}{\frac{3x^3}{y} + ax} = \frac{ay^2}{3x^3 + axy}$.

$$\therefore AB = \frac{ay^2}{3x^2} = \frac{ay}{3x} \times \frac{y}{x}$$

$$= \frac{a}{3} \times \frac{y}{x} = AC = \frac{a}{3}.$$

Taking, therefore, $AB = AC = \frac{a}{3}$, and joining CB, &c. we shall get the asymptote required.

Other curves having asymptotes are, the *Cissoid* $\left(y^2 = \frac{x^3}{a-x}\right)$, *Conchoid* $(a+x^2).(b^2-x^2)=x^2y^2$, the *Witch* $\left(y = \frac{a\sqrt{ax-x^2}}{x}\right)$. Also the curves whose equations are $y^3 = ax^2 + x^3$, and $ax^4 - by^4 + cxy = 0$.

Another method of determining asymptotes, is *by finding from the equation of the tangent*

$$y' - y = \frac{\dot{y}}{\dot{x}} . (x' - x)$$

*for what values of $x$ its ordinates become infinite.*

Ex. In the Cissoid we have (page 311)

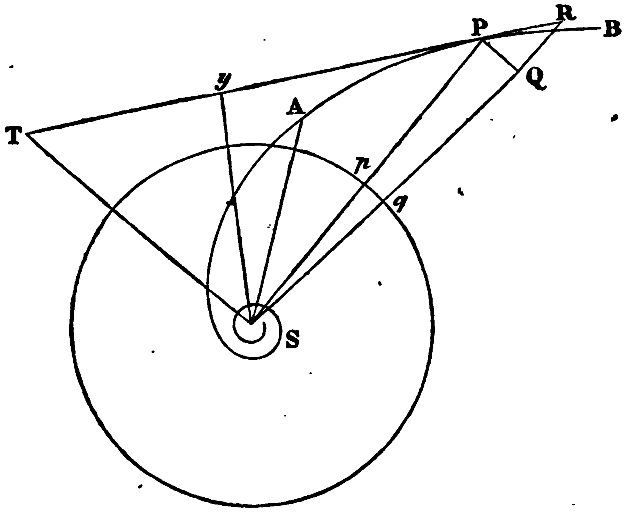
$$y' = \frac{\sqrt{x}}{2.(a-x)^{\frac{3}{2}}} \left(\overline{3a-2x}\,.\,x' - ax\right)$$

which becoming infinite, when $x=a$, coincides with the tangent, or is an asymptote.

Other applications will readily suggest themselves.

(4.) *Given the Equation between the Radius Vector ($\rho$), and the Angle described ($\theta$), of a Curve, to draw a Tangent at any Point of it.*

Let S P be any position of the *radius vector* of the curve A B whose pole is S. Also let S T be drawn at right angles to S P, meeting the tangent T P R in T,



and with S as center, and radius = 1, describe the circle A $p$ cutting the curve in A and S P in $p$.

Then, since the point P has two motions, one in the direction S P, and the other perpendicular to it, arising from the angular motion of S P, if the indefinitely small space P R, uniformly described and measuring the velocity of P in the curve, be resolved into P Q, Q R perpendicular and parallel to S P respectively, P Q, Q R will represent the velocities of P in those directions, or the fluxions of a circle whose radius is S P, and of S P.

Let S P = $\rho$, and A $p$ = $\theta$.

Then $pq = \dot{\theta}$, and we have $PQ = \frac{SP}{Sp} \times pq = \rho\dot{\theta}$.

Also Q R : P Q :: S P : S T, whence the subtangent

$ST = \frac{\rho^2\dot{\theta}}{\dot{\rho}}$.

Ex. 1. *To draw a Tangent to a Curve whose Equation is*

$$\theta = \frac{\rho^n}{a^n}$$

$$\frac{\dot{\theta}}{\dot{\rho}} = \frac{n\rho^{n-1}}{a^n}, \therefore \frac{\rho^2\dot{\theta}}{\dot{\rho}} = \frac{n\rho^{n+1}}{a^n}.$$

Take, therefore, S T $= \frac{n\rho^{n+1}}{a^n}$ and join T P: T P will be the tangent required.

When $n=1$, the curve is the *Spiral* of Archimedes.

When $n=-1$, it is the *Reciprocal* Spiral.

When $n=-2$, it is *Cote's Lituus.*

Ex. 2. *In the Involute of a Circle*

$$\dot{\theta} = \dot{\rho} \,.\, \frac{\sqrt{\rho^2 - a^2}}{\rho}$$

Hence, *subtan.* $= \frac{\rho^2\dot{\theta}}{a\dot{\rho}} = \frac{\rho\sqrt{\rho^2 - a^2}}{a}.$

Ex. 3. *In the Conchoid*

$$\dot{\theta} = \frac{b^2\dot{\rho}}{(\rho-a)\sqrt{\rho^2-2a\rho+a^2-b^2}}.$$

Hence, *subtan.* $= \frac{\rho^2\dot{\theta}}{\dot{\rho}} = \frac{b\rho^2}{(\rho-a)\sqrt{\rho^2-2a\rho+a^2-b^2}}.$

Other examples are the *Conic Sections* referred to the *focus*, a *circle* whose pole is in the circumference, and the curves whose equations are, $\theta = \frac{1}{\sqrt{a\rho - \rho^2}},$ $\rho^2 = a\theta - \theta^2$, $\theta = a\rho + b\rho^2 + c\rho^3 +$ &c.

In this place we might dilate upon the conditions of contact, &c. of any two curves referred to the same pole, and deduce conclusions similar to those for curves referred to an axis; but matter of greater importance claims our attention.

(5). *To find the Asymptotes of Curves referred to a Pole.*

Compute the subtangent $\frac{\rho^2\dot{\theta}}{\dot{\rho}}$ for the infinite value of $\rho$, and find the corresponding value of $\theta$. Draw a perpendicular to that radius vector which passes through the extremity of this value of $\theta$, and measure upon it the value of the subtangent. Then the perpendicular at the extremity of this subtangent will be the asymptote required.

If when $\rho$ is infinite, $\theta$ is finite, or $= 0$, the curve admits an asymptote.

If when $\rho$ is infinite, $\theta$ is infinite, there is no asymptote.

If when $\theta$ is infinite, $\rho$ is finite, the curve admits a circular asymptote, the radius of the circle being that finite value of $\rho$.

Ex. 1. Let $\theta = \frac{a^n}{\rho^n}$.

Then, since $\theta$ is not infinite when $\rho$ is, the curve admits an asymptote, and the subtangent

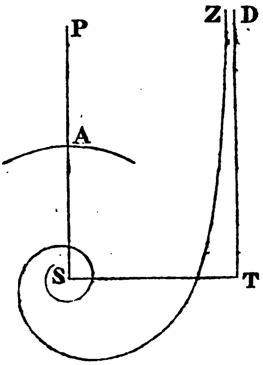
$$ST = \frac{-na^n}{\rho^{n-1}} = 0$$

when $\rho$ is infinite, for all values of $n$ except 1, and $\theta$ then $= 0$.

When $n = 1$, $ST = -a$.

Hence, supposing $\theta$ to begin at A, the infinite value of $\rho$ will pass through A, and S T, perpendicular to S.P, being put $= a$, then T D, perpendicular to S T, will be the asymptote.

This is the *Reciprocal Spiral*

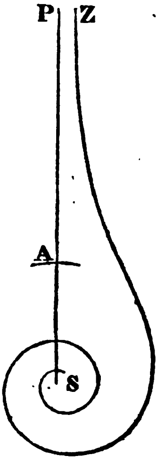


Let $n = 2$.

Then $ST = 0$, and the origin being at A, $\theta = 0$, and the asymptote coincides with S P.

This is the *Lituus*.

When $n$ is any number greater than 2, S P is still the asymptote.



Ex. 2. In the *Conchoid*, the equation is

$$\textit{secant}\ \theta = \rho - a,$$

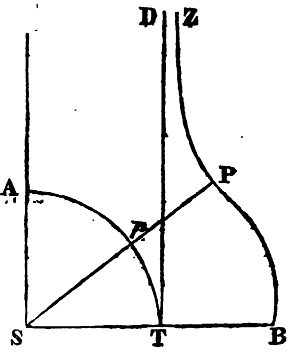
$$\text{and } \therefore ST = \frac{b\rho^2}{(\rho - a)\sqrt{\rho^2 - 2a\rho + a^2 - b^2}}$$

$$= b, \text{ when } \rho \text{ is infinite.}$$

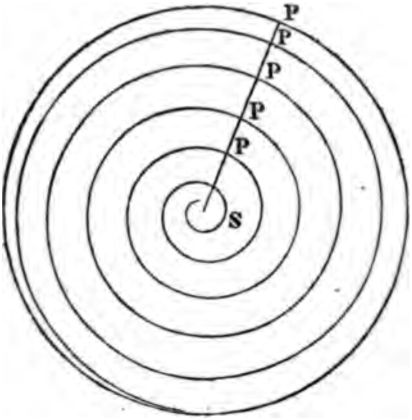
and since $sec.\ \theta = \infty$, when $\rho = \infty$

$$\theta = 90^\circ.$$

Hence, if T be the beginning of $\theta$, T A a quadrant, and $ST = b$, the perpendicular T D will be the asymptote.



Ex. 3. Let $\theta = \dfrac{1}{\sqrt{a\rho - \rho^2}}$.



When $\theta = \infty$, $\rho = a$.

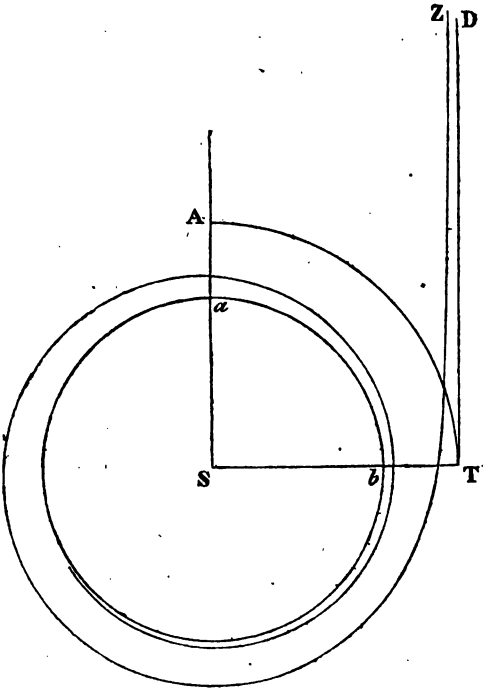
Also $\rho$ cannot exceed $a$, because then $\theta$ would be imaginary.

$\therefore$ a circle whose radius is $a$ is an asymptote to the spiral.

Also $\rho$ makes an infinite number of revolutions before it terminates in the center. For when $\rho = 0$, $\theta = \infty$.

Ex. 4. Let $\theta = \dfrac{1}{\sqrt{\rho^2 - a\rho}}$.

Then, since $\rho$ must always be greater than $a$, and yet may approach it nearer than by any assignable difference, the circle $ab$ described with the center S and radius $= a$ will be an asymptote to the spiral.



Again, the subtangent

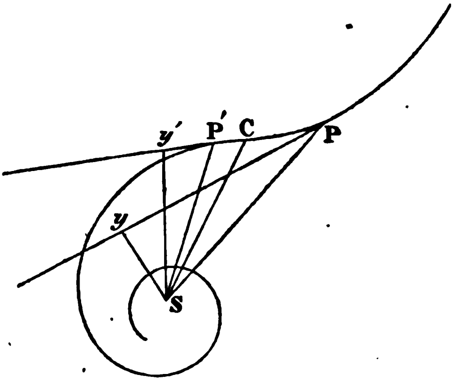
$$ST = \frac{\frac{a}{2}\rho^2 - \rho^3}{(\rho^2 - a\rho)^{\frac{3}{2}}} = -1, \text{ when } \rho = \infty.$$

And $\theta$ then $= 0$.

Hence, A being the beginning of $\theta$, and ST $= 1$, TD perpendicular to ST, will be an asymptote to the curve.

SECT. IV. Simpson, in this section, having omitted to treat of Spirals and those curves which are referred to a center or pole, we shall endeavour to supply the defect. See page 83.

### (1). *To determine the Points of contrary Flexure in Curves defined by Polar Co-ordinates.*



Let C be the point of contrary flexure. Then supposing P C convex towards S, and C P′ S concave, it is evident that the perpendicular upon the tangent S y increases from P to C, and afterwards decreases. At C therefore the perpendicular is a maximum, and its fluxion = 0.

Let S$y$ = $p$.

Now from similar triangles (fig. page 316)

$$Sy : Py :: PQ : QR,$$

or retaining our former notation,

$$p : \sqrt{\rho^2 - p^2} :: \rho \times \dot{\theta} : \dot{\rho}$$

$$\therefore p^2 : \rho^2 :: \rho^2\dot{\theta}^2 : \rho^{\dot{2}} + \rho^2\dot{\theta}^2$$

Hence $p = \dfrac{\rho^2\dot{\theta}}{\sqrt{\dot{\rho} + \rho^2\dot{\theta}^2}} = \dfrac{\rho^2}{\sqrt{\dfrac{\dot{\rho}^2}{\dot{\theta}^2} + \rho^2}}$

and taking the fluxions on the supposition that $\dot{\theta}$ is constant, we obtain

$$\frac{\dot{p}}{\dot{\rho}} = \frac{\dfrac{2\rho\dot{\rho}^2}{\dot{\theta}^2} - \dfrac{\rho^2\ddot{\rho}}{\dot{\theta}^2} + \rho^3}{\left(\rho^2 + \dfrac{\dot{\rho}^2}{\dot{\theta}^2}\right)^{\frac{3}{2}}}.$$

But at the point of contrary flexure $\dot{p} = 0$; hence then

$$\frac{2\rho\dot{\rho}^2}{\dot{\theta}^2} - \frac{\rho^2\ddot{\rho}}{\dot{\theta}^2} + \rho^3 = 0$$

$$\text{or } \frac{\ddot{\rho}}{\dot{\theta}^2} - \frac{2}{\rho}\cdot\frac{\dot{\rho}^2}{\dot{\theta}^2} = \rho \ldots\ldots\ldots\ldots (f)$$

is the equation of condition for a point of *contrary flexure.*

Ex. 1. *Required the Points of contrary Flexure in the Curve whose Equation is* $\theta = \frac{1}{\rho^n}$.

$$\frac{\dot{\theta}}{\dot{\rho}} = -\frac{n}{\rho^{n+1}}, \therefore \frac{\dot{\rho}}{\dot{\theta}} = -\frac{\rho^{n+1}}{n}$$

$$\text{Hence } \frac{\ddot{\rho}}{\dot{\theta}^2} = -\frac{n+1}{n}\cdot\frac{\rho^n}{n}\times\frac{\dot{\rho}}{\dot{\theta}}$$

$$= \frac{n+1}{n^2}\cdot\rho^{2n+1}$$

Therefore, by substitution in equation *(f)* we get

$$\frac{n+1}{n^2}\rho^{2n+1} - \frac{2}{\rho}\cdot\frac{\rho^{2n+2}}{n^2} = \rho$$

$$\left.\begin{aligned}\text{Hence } \rho &= \left(\frac{n^2}{n-1}\right)^{\frac{1}{2n}}\\ \text{and } \theta &= \frac{1}{\rho^n} = \frac{\sqrt{n-1}}{n}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots (g).$$

If $n$ be negative, these equations are impossible, or the spiral has no points of inflexion.

If $n=1$, $\rho$ is infinite, and curve has no inflexion.

$$\left.\begin{aligned}\text{If } n = 2,\ \rho &= 4^{\frac{1}{4}} = \pm\sqrt{2}\\ \text{and } \theta &= \pm\frac{1}{2}\end{aligned}\right\}$$

and all higher values of $n$ will give points of inflexion.

Ex. 2. *In the Conchoid considered as a Spiral, required the Points of inflexion.*

Here *sec.* $\theta = \rho - a$.

$$\therefore \dot{\theta} = \frac{b^2 \dot{\rho}}{(\rho - a)\sqrt{(\rho - a)^2 - b^2}} \text{ (page 280).}$$

$$\text{Hence } \frac{\dot{\rho}}{\dot{\theta}} = \frac{\rho - a}{b^2} \cdot \sqrt{\rho^2 - 2a\rho + a^2 - b^2}$$

$$\frac{\ddot{\rho}}{\dot{\theta}^2} = \frac{\rho - a}{b^4} \times (2\rho^2 - 4a\rho + 2a^2 - b^2)$$

and substituting in *(f)* we find, after the requisite reductions, that

$$\rho^3 + \frac{b^2 - 6a^2 - b^4}{2a}\rho^2 + \frac{2a^2 - 3b^2 + 4a}{2}\rho + a.(b^2 - a^2) = 0,$$

the solution of which will give the values of $\rho$ corresponding to the points of inflexion required.

In this place, if our limits would permit, we might give an exposition of the theory of singular points in general; as of *Multiple Points*, *Points of Undulation*, *Conjugate Points*, *Points of Double Undulation*, &c. &c. On this subject the reader may consult, with great advantage, *Cramer*, *Introduction à l'Analyse des Lignes Courbes*, and *A Collection of Examples of the Applications of the Differential and Integral Calculus*. By G. Peacock, A. M. &c.

SECT. V The theory of Contact and Osculation is better explained as follows.

Let two curves whose equations are $y = f(x)$, $y' = f'(x')$, ($x$ and $x'$ being measured from the same point, and along the same line), touch one another.

Then (page 307)

$$y' = y \text{ and } \frac{\dot{y}'}{\dot{x}'} = \frac{\dot{y}}{\dot{x}}.$$

Also, if we suppose $x$ and $x'$ to increase by the quantity

$h$, and the corresponding increments of $y$, $y'$ to be $k$, $k'$ respectively, Taylor's theorem will give us

$$k = \frac{\dot{y}}{\dot{x}}\, h + \frac{\ddot{y}}{\dot{x}^2} \cdot \frac{h^2}{1.2} + \frac{\dddot{y}}{\dot{x}^3} \cdot \frac{h^3}{1.2.3} + \ldots\ldots$$

$$\text{and } k' = \frac{\dot{y}'}{\dot{x}'}\, h + \frac{\ddot{y}'}{\dot{x}'^2} \cdot \frac{h^2}{1.2} + \frac{\dddot{y}'}{\dot{x}'^3} \cdot \frac{h^3}{1.2.3} + \ldots\ldots$$

and $\therefore$ the distance between the curves

$$k - k' = \left(\frac{\dot{y}}{\dot{x}} - \frac{\dot{y}'}{\dot{x}'}\right) h + \left(\frac{\ddot{y}}{\dot{x}^2} - \frac{\ddot{y}'}{\dot{x}'^2}\right) \frac{h^2}{1.2} + \&c.$$

measured along the ordinate corresponding to the abscissa $x + h$; or the deflection from the tangent, is the less in proportion, as there are more of the equations

$$\left.\begin{aligned} \frac{\dot{y}}{\dot{x}} &= \frac{\dot{y}'}{\dot{x}'} \\ \frac{\ddot{y}}{\dot{x}^2} &= \frac{\ddot{y}'}{\dot{x}'^2} \\ \&c. &= \&c. \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (g)$$

that is, the contact is so much the more intimate. It is said to be of the first, second, third, &c. orders, according as one, two, three, &c. of these equations obtain.

Hence we reduce the whole theory of *Osculation* to the solution of the following problem.

*Given the nature of a Curve to find its dimensions* (i. e. *its indeterminate Constants*) *so that it may touch a given Curve in a given Point more intimately than any other of the same kind.*

(1.) Let $x'$, $y'$ be the co-ordinates of the tangent curve, and first, let it be a straight line.

Then, since $\frac{\dot{y}'}{\dot{x}'}$ is constant, and $\frac{\ddot{y}'}{\dot{x}'^2}$, $\frac{\dddot{y}'}{\dot{x}'^3}$, &c. all vanish, and $\frac{\ddot{y}}{\dot{x}^2}$, $\frac{\dddot{y}}{\dot{x}^3}$, &c. do not vanish, except when

$(x, y)$ is also a straight line, only one of the equations $(g)$ obtain.

*Hence one straight line cannot touch a curve more closely than another.*

When $(x, y)$ and $(x', y')$ are both straight lines, then an infinite number of equations obtain, and the lines actually coincide.

(2). Let $(x', y')$ be a circle whose general equation is

$$(x' - \alpha)^2 + (y' - \beta)^2 = R^2$$

$R$ being the radius, and $(\alpha, \beta)$ the co-ordinates of its center referred to the line of abscissa of the given curve $(x, y)$.

$$\left.\begin{aligned} \text{Then } (x'-\alpha)+(y'-\beta)\,.\,\frac{\dot{y}'}{\dot{x}'} &= 0 \\ \text{and } 1 + \frac{\dot{y}'^2}{\dot{x}'^2} + (y'-\beta)\,.\,\frac{\ddot{y}'}{\dot{x}'^2} &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (h)$$

Hence obtaining $y' - \beta$ and $x' - \alpha$, and substituting them in the given equation, we get

$$R = \frac{\left(1+\dfrac{\dot{y}'^2}{\dot{x}'^2}\right)^{\frac{3}{2}}}{\dfrac{\ddot{y}'}{\dot{x}'^2}}.$$

But at the point of contact

$$\frac{\dot{y}'}{\dot{x}'} = \frac{\dot{y}}{\dot{x}}$$

$$\text{and } \frac{\ddot{y}'}{\dot{x}'^2} = \frac{\ddot{y}}{\dot{x}^2}$$

$$\therefore R = \frac{\left(1 + \dfrac{\dot{y}^2}{\dot{x}^2}\right)^{\frac{3}{2}}}{\dfrac{\ddot{y}}{\dot{x}^2}} \ldots\ldots\ldots (i)$$

which completely determines the radius of the circle which touches the curve $(x, y)$ more intimately than any other; because no other circle will require more of the equations *(g)* to obtain.

This is a contact of the *second order*.

The circles of *simple contact* are innumerable; for, from the equation of the circle, the first of *(h)*, and the conditions of *simple* contact $\frac{\dot{y}}{\dot{x}} = \frac{\dot{y}'}{\dot{x}'}$, $y = y'$, we readily obtain

$$\alpha = x + \frac{R \,.\, \frac{\dot{y}}{\dot{x}}}{\sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}}, \text{ and } \beta = y - \frac{R}{\sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}}$$

$$\left.\begin{aligned} \text{whence } R &= (\alpha - x)\sqrt{1 + \frac{\dot{x}^2}{\dot{y}^2}} \\ \text{or } R &= (y - \beta)\sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}} \end{aligned}\right\} \ldots\ldots\ldots\ldots (k)$$

which, since $\alpha$ and $\beta$ are indeterminate, must have innumerable values.

Equating these values of $R$, we get

$$\beta - y = -\frac{\dot{x}}{\dot{y}} \,.\, (\alpha - x) \;\ldots\ldots (l)$$

which denotes that the locus of the points whose co-ordinates are $(\alpha, \beta)$, or the centers of the circles of *simple contact*, is the *normal* of the given curve at the point of contact.

Proceeding as above, we might determine the degrees of contact of other curves. We should find that the ellipse, having four constants in its general equation, has three orders of contact; and in general, that an osculating curve having $(n+1)$ constants in its general equation, can only be determined in *its dimensions* by $n$ of the equations *(g)* obtaining, and therefore, that it is susceptible of a contact of the $n^{th}$ order. This we must leave to the reader.

Geometers have agreed to adopt the circle, from its regularity of curvature and general simplicity, as the measure of curvature of other curves; and that circle which touches a curve most intimately at any point, is called the *circle of curvature*, and its radius, the *radius of curvature of that point.*

$$\text{Hence } R = \frac{\left(1 + \frac{\dot{y}^2}{\dot{x}^2}\right)^{\frac{3}{2}}}{\frac{\ddot{y}}{x^2}} \text{ ............ } (m)$$

the radius of curvature [p. 72.]

In the above expression $\dot{x}$ is constant, but, $z$ being the arc corresponding to $x, y$,

$$R = \frac{\dot{z}^3}{\left(\frac{\dot{y}}{\dot{x}}\right)^{\cdot} \dot{x}^2} \text{, and making } \dot{x} \text{ vary, we have}$$

$$\left(\frac{\dot{y}}{\dot{x}}\right)^{\cdot} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}$$

$$\therefore R = \frac{\dot{z}^3}{\ddot{y}\dot{x} - \ddot{x}\dot{y}} \text{ ............ } (n) \text{ [p. 74.]}$$

which is the complete expression for $R$.

(3.) The above method will apply with the same success in Polar Curves. We prefer, however, deducing the expression for $R$ in this case, as follows:

If $\rho$ be the *radius vector* of a curve and $\theta$ its inclination to the line along which $x$ would be measured, we have

$$\rho = \sqrt{x^2 + y^2}$$

$$x = \rho \,.\, cos.\, \theta \text{ and } y = \rho \,.\, sin.\, \theta$$

and from the equation of the circle of curvature, and the conditions of contact [p. 326] we get, by substitution,

$$(\rho\, cos.\, \theta - \alpha)^2 + (\rho\, sin.\, \theta - \beta)^2 = R^2 \text{ ........ } (p)$$

and taking the fluxions twice with regard to $\rho$ and $\theta$, we get

$$(\rho \,.\, cos.\, \theta - \alpha) \left(\frac{\dot\rho}{\dot\theta} cos.\, \theta - \rho\, sin.\, \theta\right) + (\rho\, sin.\, \theta - \beta) \times$$

$$\left(\frac{\dot\rho}{\dot\theta}\, sin.\, \theta + \rho\, cos.\, \theta\right) = 0, \text{ and } \left(\frac{\dot\rho}{\dot\theta}\, cos.\, \theta - \rho\, sin.\, \theta\right)^2$$

$$+ (\rho\, cos.\, \theta - \alpha) \times \left(\frac{\ddot\rho}{\dot\theta^2} cos.\, \theta - 2\, \frac{\dot\rho}{\dot\theta}\, sin.\, \theta - \rho\, cos.\, \theta\right) +$$

$$\left(\frac{\dot\rho}{\dot\theta}\, sin.\, \theta + \rho\, cos.\, \theta\right)^2 + (\rho\, sin.\, \theta - \beta) \,.\, \left(\frac{\ddot\rho}{\dot\theta^2}\, sin.\, \theta +\right.$$

$$\left. 2\, \frac{\dot\rho}{\dot\theta}\, cos.\, \theta - \rho\, sin.\, \theta\right) = 0\,;$$ whence obtaining $\rho\, sin.\, \theta - \beta$, and $\rho\, cos.\, \theta - \alpha$, and substituting them in equation *(p)* we finally obtain

$$R = \frac{\left(\rho^2 + \frac{\dot\rho^2}{\dot\theta^2}\right)^{\frac{3}{2}}}{\rho^2 + 2\, \frac{\dot\rho^2}{\dot\theta^2} - \rho \,.\, \frac{\ddot\rho}{\dot\theta^2}} \quad \text{.................} \quad (q_1)$$

(4.) The *evolute* is evidently the locus of the centers of the circles of curvature for the different points of the given curve. Its co-ordinates are therefore $\alpha$ and $\beta$, and its equation will be determined very easily by substituting for $y$, $\frac{\dot y}{\dot x}$, $\frac{\ddot y}{\dot x^2}$, deduced from the equation of the curve in terms of $x$, in the equations of contact [see *h*].

$$\left.\begin{array}{l} (x - \alpha) + (y - \beta) \,.\, \frac{\dot y}{\dot x} = 0 \\ \text{and } 1 + \frac{\dot y^2}{\dot x^2} + (y - \beta) \,.\, \frac{\ddot y}{\dot x^2} = 0 \end{array}\right\} \quad \text{............} \quad (r)$$

and hence eliminating $x$.

We shall thus obtain the relation between $\alpha$ and or the equation of the *evolute*.

*The evolute of every algebraical curve, being equal in length, to the difference of the radii of curvature corresponding to its extremities, is always rectifiable.*

We have not much room for examples in this theory. Take the following:

Ex. 1. *Let the curve be the rectangular hyperbola referred to its asymptotes*, or $x y = a^2$.

Then we readily find

$$R = \frac{-x^3}{2a^2}\left(1 + \frac{a^4}{x^4}\right)^{\frac{3}{2}}.$$

and from equations $(r)$ and that of the curve

$$\alpha = \frac{3x}{2} + \frac{a^4}{2x^3}, \text{ and } \beta = \frac{3a^2}{2x} + \frac{x^3}{2a^2}$$

Hence, to eliminate $x$, we should have, by the usual method, to solve an equation of the third degree. An artifice, however, will simplify the matter.

$$\text{Since } \alpha = \frac{a}{2} \times \left(3\frac{x}{a} + \frac{a^3}{x^3}\right), \text{ and } \beta = \frac{a}{2}\cdot\left(\frac{3a}{x} + \frac{x^3}{a^3}\right)$$

$$\left.\begin{aligned} \therefore \alpha + \beta &= \frac{a}{2}\cdot\left(\frac{a}{x} + \frac{x}{a}\right)^3 \\ \text{and } \alpha - \beta &= \frac{a}{2}\cdot\left(\frac{a}{x} - \frac{x}{a}\right)^3 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore \frac{a}{x} + \frac{x}{a} &= \left(\frac{2}{a}\right)^{\frac{1}{3}}\cdot(\alpha + \beta)^{\frac{1}{3}} \\ \text{and } \frac{a}{x} - \frac{x}{a} &= \left(\frac{2}{a}\right)^{\frac{1}{3}}\cdot(\alpha - \beta)^{\frac{1}{3}} \end{aligned}\right\}$$

$$\text{Hence } \frac{2x}{a} = \left(\frac{2}{a}\right)^{\frac{1}{3}}\cdot\left\{(\alpha + \beta)^{\frac{1}{3}} - (\alpha - \beta)^{\frac{1}{3}}\right\}$$

$$\text{and } \frac{2x}{a} = \left(\frac{a}{2}\right)^{\frac{1}{3}} \times \frac{1}{(\alpha + \beta)^{\frac{1}{3}} + (\alpha - \beta)^{\frac{1}{3}}}$$

and $\therefore \left(\frac{a^2}{4}\right)^{\frac{1}{3}} = (\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}}$, the equation to the *evolute*.

Ex. 2. *In the ellipse* $\left(y^2 = \frac{b^2}{a^2} \cdot \overline{a^2 - x^2}\right)$, we shall find

$$R = \frac{\left(a^2 - \frac{a^2 - b^2}{a^2} x^2\right)^{\frac{3}{2}}}{ab},$$

and the equation to the *evolute*,

$$(a \cdot \alpha)^{\frac{2}{3}} + (b \cdot \beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

Ex. 3. *In the Trisectrix the equation is* $\rho = 2 \cos. \theta \pm 1$; and we easily find by equation ($q$) that

$$R = \frac{(5 \pm 4 \cos. \theta)^{\frac{3}{2}}}{9 \pm 6 \cos. \theta}$$

SECT. VI.—This Section, treating of the fluents of *algebraic* functions only, is imperfect; in investigating the areas, &c. of *all sorts* of curves, we may have occasion to find the fluents also of *exponential*, *logarithmic*, and *circular* functions.

(1.) *To find the fluent of* $a^x \dot{x}$.

Let $a^x = u$

Then $\dot{u} = a^x \dot{x} . l. a$ [p. 277.]

$\therefore \int a^x \dot{x} = \frac{u}{l. a} + c = \frac{a^x}{l. a} + c$, $c$ being supposed the constant which vanished by taking the fluxion, and $\int$ denoting the fluent of the quantity which immediately follows it.

*Generally, required the fluent of $U \dot{x}$, where $U$ is any algebraic function of $a^x = u$, whatever.*

Since $\dot{x} = \dfrac{\dot{u}}{ul.a}$ as above, we have

$\int U \dot{x} = \displaystyle\int \frac{U\dot{u}}{ul.a}$, which is reduced to an *algebraic* form.

(2.) *To find the fluent of $x . a^x\dot{x}$, which is partly algebraic and partly exponential.*

Since $(vw)^{\cdot} = \dot{v}w + \dot{w}v$

and $\therefore \int \dot{v}w = vw - \int \dot{w}v$, we have

$$\int a^x\dot{x} \times x = \frac{a^x}{l.a} \times x - \int \dot{x}\ \frac{a^x}{l.a} = \frac{xa^x}{l.a} - \frac{a^x}{(la)^2} + c$$

the fluent required.

*And generally, required the fluent of $X a^x\dot{x}$, where $X$ is any algebraic function of $x$.*

Let $X_1 = \dfrac{\dot{X}}{\dot{x}}$,

$X_2 = \dfrac{\ddot{X}}{\dot{x}^2}$

&c. = &c.

Then since $\int X \times a^x\dot{x} = \dfrac{a^x}{la}\, X - \dfrac{1}{la} . \displaystyle\int \frac{\dot{X}}{\dot{x}}\, a^x\dot{x} =$ $\dfrac{a^x}{la}\, X - \dfrac{1}{la} . \int X_1\, a^x\, \dot{x}$, if the process be continued we shall obtain

$$\int Xa^x\dot{x} = \frac{Xa^x}{la} - \frac{X_1a^x}{(la)^2} + \frac{X_2\, a^x}{(la)^3} - \&c.$$

$$e.g. \int x^n a^x\dot{x} = \frac{a^x}{la} . \left\{ x^n - \frac{nx^{n-1}}{la} + \ldots\ldots \pm \frac{n.(n-1)\ldots 2.1}{(la)^n} \right\}$$

when $n$ is an integer.

This method will always reduce direct powers of $x$ or its functions.

When the powers are inverse or in the denominator, we must make

$$X_1 = \int X \dot{x}$$
$$X_2 = \int X_1 \dot{x}$$
$$\&c. = \&c. \text{ and}$$
$$\int X\dot{x} \times a^x = X_1 a^x - la \int X_1 \dot{x} \,.\, a^x$$

continued, will give

$$\int X\dot{x} \times a^x = X_1 a^x - la\, X_2 a^x + (la)^2 \,.\, X_3 a^x - \&c.$$

$$e\,.\,g\,.\int \frac{a^x \dot{x}}{x^n} = -\, a^x \,.\, \left\{ \frac{1}{(n-1)\, x^{n-1}} - \frac{l\,.\,a}{(n-1)\,(n-2)\, x^{n-2}} \right.$$

$$\cdots\cdots\cdots \frac{(l\,.\,a)^{n-1}}{(n-1)\,\ldots\, 2\,.\,1} \int \frac{a^x \dot{x}}{x}\,.$$

(3.) *To find the fluent of* $\dot{x}\, l\,.\, x$.

Since $(lx)^{\cdot} = \dfrac{\dot{x}}{x}$ [p. 277.]

$$\int \dot{x} \times lx = xlx - \int \frac{\dot{x}}{x} \times x$$
$$= xlx - x + c,$$

and generally, *required the fluent of* $\dot{x} f\,.\,(l\,.\,x)$. where $f$ denotes an algebraic function.

Let $lx = u$

Then $x = e^u$

and $\dot{x} = \dot{u}\, e^u$ [p. 277.]

Hence $\int \dot{x} f.(lx) = \int e^u \dot{u} f(u)$ which may be found by the general process explained in the preceding article.

(4.) *To find the fluent of* $X \dot{x}\, l\,.\, x$, *where* $X$ *is any Algebraic Function of* $x$.

Here $\int X\dot{x} \,.\, lx = lx \int X\dot{x} - \int \left\{ \frac{\dot{x}}{x} \times \int X\dot{x} \right\}$

which will be algebraic if $\int X\dot{x}$ come out in that form.

Ex. 1. Let $X = x^n$.

Then $\int x^n \dot{x} lx = \frac{x^{n+1}}{n+1} lx - \int x^n \dot{x} = \frac{x^{n+1}}{n+1} \times (lx - 1) + C.$

*And generally, to find the Fluent of* $X\dot{x}(l.x)^n$.

$$\begin{aligned} \text{Let } X_1 &= \int X\dot{x} \\ X_2 &= \int X_1\dot{x} \\ \text{\&c.} &= \text{\&c.} \end{aligned}$$

Then, by continuing the above process, we shall get

$$\int X\dot{x}(lx)^n = X_1(lx)^n - nX_2(lx)^{n-1} + n\,.\,(n-1) \times X_3(lx)^{n-2} - \text{\&c.}$$

(5.) *To find the Fluent of* $\frac{X\dot{x}}{(lx)^n}$.

$$\begin{aligned} \text{Let } X_1 &= \frac{(Xx)^{\cdot}}{\dot{x}} \\ X_2 &= \frac{(X_1x)^{\cdot}}{\dot{x}} \\ \text{\&c.} &= \text{\&c.} \end{aligned}$$

$$\begin{aligned} \text{Then } \int \frac{X\dot{x}}{(lx)^n} &= \int Xx \times \frac{\dot{x}}{x}(lx)^{-n} \\ &= \frac{-Xx}{(n-1)(lx)^{n-1}} + \int \frac{(Xx)^{\cdot}}{(n-1)(lx)^{n-1}} \\ &= \frac{-Xx}{(n-1)(lx)^{n-1}} + \int \frac{X_1\dot{x}}{(n-1)(lx)^{n-1}}, \end{aligned}$$

for $\frac{\dot{x}}{x} = (l.x)^{\cdot}$

Hence, by continuing the process, we obtain

$$\int \frac{X\dot{x}}{(lx)^n} = \frac{-Xx}{(n-1)\,(lx)^n} - \frac{X_1\,x}{(n-1).(n-2)(lx)^{n-2}} -$$

&c., constantly diminishing the negative index of $lx$.

It is worthy of remark, that the general form $\frac{\dot{x}}{x} \times f.\,(lx)$ being $= (lx)^{\cdot} \times f.\,(lx)$, may be always integrated.

(6.) *To find the fluent of $\dot{x}$ sin. $x$, $\dot{x}$ tan. $x$, $\dot{x}$ sec. $x$, and $\dot{x}$ vers. $x$.*

First $\int \dot{x} \sin. x = \int -(\cos. x)^{\cdot}$ (see page 279).

$$= -\cos. x + c \ldots\ldots\ldots\ldots (a)$$

Hence $\int \dot{x} \cos. x = \int \dot{x} \sin. \left(\frac{\pi}{2} - x\right) = \int -\left(\frac{\pi}{2} - x\right)^{\cdot} \sin. \left(\frac{\pi}{2} - x\right) = \cos. \left(\frac{\pi}{2} - x\right) + c = \sin. x + c \ldots\ldots\ldots\ldots\ldots (b)$

Again $\int \dot{x} \tan. x = \int \frac{\dot{x} \sin. x}{\cos. x} = \int \frac{-(\cos. x)^{\cdot}}{\cos. x} = -l.(\cos. x) + c.$

$\therefore \int \dot{x} \tan. x = \int \dot{x}. \frac{\sin. x}{\cos. x} = \int \frac{\dot{x}}{\cot. x} = -l. \cos. x + c \ldots (c)$

Hence $\int \dot{x} \cot. x = \int \dot{x} \tan. \left(\frac{\pi}{2} - x\right) = -\int \left(\frac{\pi}{2} - x\right)^{\cdot} \tan. \left(\frac{\pi}{2} - x\right) = l.\cos. \left(\frac{\pi}{2} - x\right) + c = l.(\sin. x) + c$

$\therefore \int \dot{x} \cot. x = \int \dot{x} \frac{\cos. x}{\sin. x} = \int \frac{\dot{x}}{\tan. x} = l. \sin. x + c \ldots . (d)$

Again $\int \dot{x} \sec. x = \int \frac{\dot{x}}{\cos. x} = \int \frac{\dot{x} \cos. x}{\cos.^2 x} = \int \frac{(\sin. x)^{\cdot}}{1 - \sin.^2 x}$

$= \int \frac{1}{2} \frac{(\sin. x)^{\cdot}}{1 + \sin. x} + \int \frac{1}{2} \frac{(\sin. x)^{\cdot}}{1 - \sin. x} = l. \sqrt{1 + \sin. x}$

$-l.\sqrt{1-\sin.x}+c=l.\sqrt{\frac{1+\sin.x}{1-\sin.x}}+c=$
$l.\tan.\left(45+\frac{x}{2}\right)+c$

$\therefore\ \int\dot{x}\sec.x=\int\frac{\dot{x}}{\cos.x}=l.\tan.\left(45+\frac{x}{2}\right)+c\ldots\ldots(e)$

Hence $\int\dot{x}\,\mathrm{cosec}.x=\int\frac{\dot{x}}{\sin.x}=l.\tan.\frac{x}{2}+c\ldots\ldots(f)$

Finally $\int\dot{x}\,\mathrm{vers}.x=\int\dot{x}.(1-\cos.x)=\int\dot{x}-\int\dot{x}\cos.x=x-\sin.x+c\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (g)$

Hence $\int\dot{x}$ coversed $\sin.x=\int\dot{x}\,\mathrm{vers}.\left(\frac{\pi}{2}-x\right)=-\int\left(\frac{\pi}{2}-x\right)^{\cdot}\mathrm{vers}.\left(\frac{\pi}{2}-x\right)=-\frac{\pi}{2}+x+\sin.\left(\frac{\pi}{2}-x\right)+c=x-\frac{\pi}{2}+\cos.x+c\ldots\ldots\ldots\ldots\ (h)$

(7.) *To find the Fluent of* $\dot{x}\sin.^{n}x\times\cos.^{m}x=F$.

First let us reduce $n$.

By the form $\int\dot{v}w=vw-\int\dot{w}v$, we have
$F=\int\dot{x}\sin.x.\cos.^{m}x\times\sin.^{n-1}x$
$=-\frac{1}{m+1}.\cos.^{m+1}x\times\sin.^{n-1}x+\frac{n-1}{m+1}\times$
$\int\dot{x}\cos.^{m+2}x.\sin.^{n-2}x.$

But $\cos.^{m+2}x\sin.^{n-2}x=\cos.^{m}x\sin.^{n-2}x\times(1-\sin.^{2}x)$
$=\cos.^{m}x\ \sin.^{n-2}x-\cos.^{m}x\sin.^{n}x.$

Hence $\frac{m+n}{m+1}F=-\frac{1}{m+1}.\cos.^{m+1}x\ \sin.^{n-1}x+$
$\frac{n-1}{m+1}\int\dot{x}\cos.^{m}x.\sin.^{n-2}x$, or

$F=-\frac{1}{m+n}\cos.^{m+1}x.\sin.^{n-1}x+\frac{n-1}{m+n}\times$
$\int\dot{x}\cos.^{m}x\ \sin.^{n-2}x\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (k)$

in which $n$ the index of $\sin.x$ is reduced by two.

Again to reduce $m$ we have

$$F = \int \dot{x} \cos. x \,.\, \sin.^{n} x \times \cos.^{m-1} x$$

$$= \frac{1}{n+1} \sin.^{n+1} x \cos.^{m-1} x + \frac{m-1}{n+1} \int \dot{x} \sin.^{n+2} x \times$$

$\cos.^{m-2} x$, and reducing as before, we obtain

$$F = \frac{\sin.^{n+1} x \,.\, \cos.^{m-1} x}{m+n} + \frac{m-1}{m+n} \int \dot{x} \sin.^{n} x \cos.^{m-2} x$$

.................... (*l*)

which reduces the index $m$ by 2.

Continuing these reductions we shall at length arrive at the simplest form of the fluent. When $m$ and $n$ are integers, our last form will be one of the forms in the preceding number.

(8.) *To find the fluent of* $\frac{\dot{x} \sin.^{n} x}{\cos.^{m} x} = F$.

To reduce $n$, we have

$$F = \int \dot{x} \,.\, \sin. x \,.\, \cos.^{-m} x \times \sin.^{n-1} x,$$

and proceeding as before, we get

$$F = -\frac{1}{n-m} \cdot \frac{\sin.^{n-1} x}{\cos.^{m-1} x} + \frac{n-1}{n-m} \int \frac{\dot{x} \sin.^{n-2} x}{\cos.^{m} x} \text{......} (m)$$

which reduces $m$ by two.

To reduce $m$ we have

$$F = \int \dot{x} \cos. x \sin.^{n} x \times \cos^{-m-1} x,$$

and we easily obtain

$$F = \frac{1}{m-1} \cdot \frac{\sin.^{n+1} x}{\cos.^{m-1} x} - \frac{n-m+2}{n-1} \int \frac{\dot{x} \sin.^{n} x}{\cos.^{m-2} x} \text{.........} (n)$$

which reduces $m$ by two.

Repeating these operations, the fluent may be reduced to its most simple form; and when $m$, $n$ are integers it is reducible to some one of those in No. 6.

Hence we may find the fluent of $\dot{x}\ tan.\,^{n}x = \frac{\dot{x}\ sin.\,^{n}x}{cos.\,^{n}x}$, of $\dot{x}\ sec.\,^{n}x = \frac{\dot{x}}{cos.\,^{n}x}$, and therefore of $\dot{x}\ cot.\,^{n}x$, and of $\dot{x}\ cosec.\,^{n}x$.

(9.) To find the fluent of $\frac{\dot{x}}{sin.\,^{n}x\,.\,cos^{m}x}$.

The process required for reducing $m$ and $n$ here, does not differ from that delivered in No. 9 for reducing $m$.

(10.) *To find the fluent of* $\dot{x}\ sin.^{-1}x$, $\dot{x}\ tan.^{-1}x$, $\dot{x}\ sec.^{-1}x$, *and* $\dot{x}\ vers.^{-1}x$, *where* $sin.^{-1}x$, *&c. denotes the arc whose sine is* $x$ *to rad.*$=1$ &c.

First, $\int\dot{x}\ sin.^{-1}x = x\ sin.^{-1}x - \int x\,(sin.^{-1}x)^{\cdot} = x\ sin.^{-1}x - \int\frac{x\dot{x}}{\sqrt{1-x^2}}$ (see page 279, line 8) $=$ $x.\,sin^{-1}x - \sqrt{1-x^2} + c$..............(o)

Again, $\int\dot{x}\ tan.^{-1}x = x\ tan.^{-1}x - \int\frac{x\dot{x}}{1+x^2}$ (see p. 279, No. 5) $= x\ tan.\,x^{-1} - \frac{1}{2}\,l.\,(1+x^2)+c$......(p)

Again, $\int\dot{x}\ sec.^{-1}x = x\ sec.^{-1}x - \int\frac{x\dot{x}}{x\sqrt{x^2-1}}$ (see p. 280) $= x\ sec.^{-1}x - \int\frac{\dot{x}}{\sqrt{x^2-1}}$. But $\frac{\dot{x}}{\sqrt{x^2-1}} =$

$$\frac{\dot{x}+\dfrac{x\dot{x}}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}} = \frac{(x+\sqrt{x^2-1})^{\cdot}}{x+\sqrt{x^2-1}}.$$

$\therefore \int\dot{x}\ sec.^{-1}x = x\ sec.^{-1}x - l.\,(x+\sqrt{x^2-1})$ ... (q).

Again, $\int \dot{x}\, vers.^{-1}x = x\, vers.^{-1}x - \int x\, .\, (vers.^{-1}x)^{\cdot}$. Let $vers.\ \theta = x = 1 - cos.\ \theta$, $\therefore \dot{x} = \dot{\theta}\ sin.\ \theta = \dot{\theta}\sqrt{1 - cos.^2\theta} = \dot{\theta}\sqrt{2x - x^2}$, $\therefore \dot{\theta} = (vers.^{-1}x)^{\cdot} = \dfrac{\dot{x}}{\sqrt{2x-x^2}}$.

Hence $\int \dot{x}\ vers.^{-1}x = x\ vers.^{-1}x - \int \dfrac{x\dot{x}}{\sqrt{2x-x^2}}$, and $\int \dfrac{x\dot{x}}{\sqrt{2x-x^2}} = -\int \dfrac{\dot{x}-x\dot{x}}{\sqrt{2x-x^2}} + \int \dfrac{\dot{x}}{\sqrt{2x-x^2}} = -\sqrt{2x-x^2} + vers.^{-1}x$.

$\therefore$ we finally obtain

$$\int \dot{x}\ vers.^{-1}x = (x-1)\ vers.^{-1}x + \sqrt{2x-x^2} + c \ldots\ldots(r)$$

Hence there will be no difficulty in getting the fluents of $\dot{x}\ cos.^{-1}x$, $\dot{x}\ cot.^{-1}x$, $\dot{x}\ cosec.^{-1}x$, and $\dot{x}\ covers.^{-1}x$.

(11.) *To find the fluent of* $X\dot{x} \times z^n$, where $X$ is any function of $x$, and $z$ the arc, whose sine, or cosine, or tangent &c. $= x$.

$$\text{Let } \frac{\dot{z}}{\dot{x}} = P, \text{ a function of } x.$$

$$\begin{aligned} \text{Let also } \int X\dot{x} &= X_1 \\ \int PX_1\dot{x} &= X_2 \\ \int PX_2\dot{x} &= X_3 \\ \&c. &= \&c. \end{aligned}$$

Then $\int X\dot{x}z^n = X_1 z^n - n\int XP\dot{x}z^{n-1}$, and repeating the operation, we get

$$\int X\dot{x}z^n = X_1 z^n - nX_2 z^{n-1} + n\,.\,(n-1)\ X_3 z^{n-2} - \&c. \ldots\ldots(s)$$

Having thus laid down the general principles of the subject, the reader is enabled to supply the detail himself.

SECT. VII. — Ex. 1. *To find the area of the logarithmic curve, whose equation is*

$$y = a^x.$$

$$\int y\dot{x} = \int a^x \dot{x} = \frac{a^x}{la} + c \text{ (see p. 331.)}$$

Let $x = 0$, then the area $= 0$, and $c = -\dfrac{1}{la}$.

$$\therefore \int y\dot{x} = \frac{a^x - 1}{la} = \frac{y - 1}{la}.$$

Ex. 2. *In the sinuoid whose equation is* $y = sin.\ x$, we have

$$\int y\dot{x} = \int \dot{x}\ sin.\ x = c - cos.\ x \text{ (see p. 335)}$$
$$= 1 - cos.\ x.$$

Ex. 3. *In the conchoid referred to a pole the equation is*

$\rho = b\ sec.\ \theta \pm a \ \therefore \displaystyle\int \frac{\rho^2 \dot{\theta}}{2} = \int \frac{b^2 \dot{\theta}\ sec.^2 \theta}{2} \pm \int a$[illegible]$sec.\ \theta$

$$+ \int \frac{a^2 \dot{\theta}}{2} = \frac{b^2}{2}.\ tan.\ \theta \pm ab.\ l.\ tan.\ \left(45^\circ + \frac{\theta}{2}\right) + \frac{a^2\theta}{2} + c$$

(see pp. 122, 335.)

Let the area $= 0$ when $\theta = 0$.

Then $c = \mp abl.(b)$.

$$\int \frac{\rho^2 \dot{\theta}}{2} = \frac{b^2}{2}\ tan.\ \theta \pm abl.\frac{tan.\ \left(45^\circ + \frac{\theta}{2}\right)}{b} + \frac{a^2\theta}{2}.$$

SECT. VIII.—In this Section, which treats of the rectification of curves, we may add the fluxional expression for the arc of a curve, as defined by a *radius vector and the angle at the pole.*

In figure page 157, if $Rm$ denote the fluxion of the curve $AR$ or $\dot{z}$, then $mr = (CR)^{\cdot} = \dot{\rho}$, and $Nn = (BN)^{\cdot} = \dot{\theta}$. Let $CB = 1$.

Then $Rr = Nn \times \dfrac{CR}{CN} = \dot{\theta}\rho$.

Hence $\dot{z} = \sqrt{mr^2 + Rr^2} = \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2}$.........(a)
which is another expression for the fluxion of an arc.

Ex. *Let* $\theta = \rho^n$ *be the equation of the curve.*

Then $\dot{z} = \dot{\rho}\sqrt{1 + n^2\rho^{2n}}$,
whose fluent cannot be found generally except by approximation.

When $n = 1$ or the curve is the *spiral of Archimedes*,

$$z = \int \dot{\rho}\sqrt{1 + \rho^2}$$

Let $\rho \,.\, \sqrt{1 + \rho^2} = u$

Then $\dot{u} = 2\dot{\rho}\sqrt{1+\rho^2} - \dfrac{\dot{\rho}}{\sqrt{1+\rho^2}}$

$$\therefore z = \frac{\rho\sqrt{1+\rho^2}}{2} - \frac{1}{2}\int \frac{\dot{\rho}}{\sqrt{1+\rho^2}}$$

$$= \frac{\rho\sqrt{1+\rho^2}}{2} + \frac{1}{2} \,.\, l \,.\, (\rho + \sqrt{1 + \rho^2}) + c$$ (see p. 140.)

Let the arc $= 0$, when $\theta = 0$; thence $c = 0$,

and $z = \dfrac{\rho\sqrt{1+\rho^2}}{2} + \frac{1}{2} \,.\, l \,.\, (\rho + \sqrt{1+\rho^2})$ (see p. 164.)

When $n = -1$, or the curve is the *reciprocal spiral*

$$z = \int \frac{\dot{\rho}}{\rho} \,.\, \sqrt{1 + \rho^2}$$

Put $\sqrt{1+\rho^2} = u$,

Then $\dot{u} = \dfrac{\rho\dot{\rho}}{\sqrt{1+\rho^2}} = \dfrac{\dot{\rho}}{\rho\sqrt{1+\rho^2}} \times (\rho^2 + 1 - 1)$

$$= \frac{\dot{\rho}}{\rho}\sqrt{1+\rho^2} - \frac{\dot{\rho}}{\rho\sqrt{1+\rho^2}}$$

Hence $z = \sqrt{1+\rho^2} + \displaystyle\int \frac{\dot{\rho}}{\rho\sqrt{1+\rho^2}}$.

But $\displaystyle\int \frac{\dot{\rho}}{\rho\sqrt{1+\rho^2}} = \frac{1}{2}\, l. \frac{\sqrt{1+\rho^2}-1}{\sqrt{1+\rho^2}+1}$ (see 141, where *Simpson* might have shown in the same way, that $\displaystyle\int \frac{2a\dot{x}}{x\sqrt{a^2 \pm x^2}}$ also $= l. \frac{\sqrt{a^2 \pm x^2} - a}{\sqrt{a^2 \pm x^2} + a}$).

Hence, and taking the correction on the supposition that the arc vanishes when $\rho$ does, we get

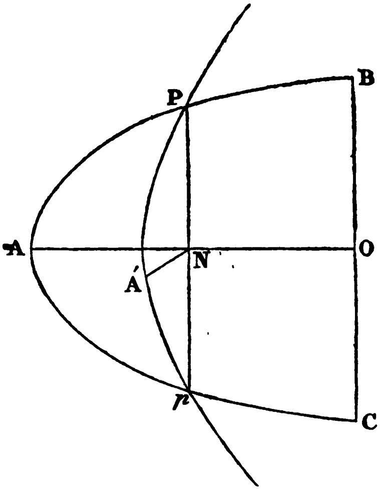
$$z = \sqrt{1+\rho^2} + l. \frac{\rho}{\sqrt{1+\rho^2}+1} - 1.$$

SECT. IX and X.—*If a solid be generated by the double motion of a curve whose plane is constantly perpendicular to that of a given curve, one in the direction of its own line of abscissæ, and the other in that of the given curve, so that the intersections of the curves may be their common ordinates, its solid content is thus obtained:*

Let $AN = x$, $PN = y$, $A'N = x'$.

Hence, since $\int y\dot{x}' =$ the area of $PA'p$, the solid content of $PApA'$ will be expressed by

$$S = \int \dot{x} \int 2y\dot{x}' = 2\int \dot{x} \int y\dot{x}'.$$



Ex. *Let* BAC *be a semicircle whose radius* BO *is* $r$, *and* P A′$p$ *a triangle, whose equation is* $y = ax'$.

Here $S = 2\int\dot{x}\int\frac{y\dot{y}}{a} = 2\int\dot{x}\,\frac{y^2}{2a} + \int c\dot{x} =$ $2\int\dot{x}\,\frac{(2rx-x^2)}{2a} + cx = \frac{rx^2}{a} - \frac{x^3}{3a} + cx + c'$. But the area B A′$p$ = 0, when $y = 0$

$\therefore c = 0$; and the solid $= 0$, when $x = 0$, $\therefore c' = 0$.

Hence $S = \frac{rx^2}{a} - \frac{x^3}{3a} = \frac{2r^3}{3a}$, when $x = r$,

$= \frac{2}{3}\cdot\frac{r^2}{a} \times r = \frac{2}{3}$ the triangular base × altitude.

Other examples will readily suggest themselves.

The expression for surfaces so generated, will evidently be $s = \int\dot{z}\int\dot{z}'$, $z$ and $z'$ being the arcs of the curves.

We might now proceed to the discussion of curves, curve surfaces, and solids, in general, as referred to three rectangular co-ordinates, that is, any how disposed in space; but it would greatly exceed the limits of this work. The reader will find all the information on this intricate subject, he can desire, in the works of *Monge*.

END OF VOL. I.

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